

THE CUSPIDAL GROUP AND SPECIAL VALUES OF L -FUNCTIONS

BY

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ABSTRACT. The structure of the cuspidal divisor class group is investigated by relating this structure to arithmetic properties of special values of L -functions of weight two Eisenstein series. A new proof of a theorem of Kubert (Proposition 3.1) concerning the group of modular units is derived as a consequence of the method. The key lemma is a nonvanishing result (Theorem 2.1) for values of the “ L -function” attached to a one-dimensional cohomology class over the relevant-congruence subgroup. Proposition 4.7 provides data regarding Eisenstein series and associated subgroups of the cuspidal divisor class group which the author hopes will simplify future calculations in the cuspidal group.

Let $X_{\Gamma/\mathbf{Q}}$ be Shimura’s canonical model over \mathbf{Q} of the complete modular curve associated to the congruence group $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, where Γ is one of the groups $\Gamma(N)$, $\Gamma_1(N)$, or $\Gamma_0(N)$. The cuspidal divisor class group $C_{\Gamma} \subseteq \mathrm{Pic}^0(X_{\Gamma})$ plays an important role in the theory of the arithmetic of the modular curve X_{Γ} , for example in Mazur’s proof of Ogg’s conjecture [13] and in the proof by Mazur and Wiles of the main conjecture of Iwasawa theory [14]. Subgroups of the cuspidal group lead to the isogenies used in the Eisenstein descent theory. These subgroups can also be used to give congruence formulas for the universal special values of the L -function of X_{Γ} which are compatible with the descent theoretic results and the conjecture of Birch and Swinnerton-Dyer [3, 5, 12, 19].

The structure of the cuspidal group has been investigated extensively by Kubert and Lang [6, 7]. Their approach is based on a study of the group of modular units.

In the present paper we propose another approach based on a study of integrality and divisibility properties of special values of L -functions of weight two Eisenstein series. Typically such an L -function is a product of two Dirichlet L -functions whose special values are closely related to the arithmetic of cyclotomic fields. Using a theorem of L. Washington [20] on the non- p -part of the class number in cyclotomic \mathbf{Z}_p -extensions we describe a method of computing subgroups of C_{Γ} which should be useful for the descent theory. The corresponding congruence formulas for universal special values of L -functions are an immediate consequence of the method.

Since Eisenstein series occur in many settings our point of view offers the prospect of generalization. Such a generalization to Hilbert-Blumenthal varieties may be an

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important step towards extending the Mazur-Wiles theorem to totally real number fields.

Our motivating observation is that the logarithmic derivative of a modular unit over Γ is a weight two Eisenstein series over Γ with integral periods on the affine curve $Y_\Gamma = X_\Gamma \setminus \{\text{cusps}\}$ (§1). Thus the problem of finding all modular units is closely related to finding the group of periods on Y_Γ of a weight two Eisenstein series E over Γ . Our starting point is Theorem 1.3 which determines the group of periods of E in terms of the special values at $s = 0$ and $s = 1$ of the L -function of E and its twists.

By relating these values to units in cyclotomic fields we give another proof of a theorem of Kubert (Proposition 3.1) which leads to a description of the group of modular units of all levels in terms of the Siegel units and square roots of certain products of Siegel units. Kubert's original proof [7] was for odd level only, though he has since modified the proof to work for arbitrary level [8].

The cuspidal group C_Γ is generated by certain subgroups $C_\Gamma(E)$ associated to weight two Eisenstein series E over Γ . In case E is an eigenfunction for the Hecke operators, the group $C_\Gamma(E)$ is preserved by the action of the Galois group and the Hecke algebra. Thus $C_\Gamma(E)$ is a natural group for the purpose of descent. Moreover, the corresponding congruence formulas for the universal special values of the L -function of X_Γ are given in terms of the special values of the L -function of E .

In §4 we introduce a natural basis of eigenforms for the space of weight two Eisenstein series over $\Gamma = \Gamma_1(N)$. For each E in this basis we describe $C_\Gamma(E)$ and the associated congruence formula. We have attempted to arrange this section to facilitate similar calculations for other eigenforms.

§4 closes with two examples on $X_0(N)$. The methods described here determine $C_{\Gamma_0(N)}(E)$ only modulo its intersection with the Shimura group. Since the Shimura group is easy to describe as a Galois module, it is often possible to show this intersection is trivial. In the general case we would need to make a study of the explicit formulas for the periods of E in terms of generalized Dedekind sums.

NOTATION. For $x \in \mathbf{C}$ we write $e(x)$ for $e^{2\pi ix}$.

The Bernoulli functions $\mathbf{B}_1, \mathbf{B}_2: \mathbf{R} \rightarrow \mathbf{R}$ are defined by

$$\mathbf{B}_1(x) = x - [x] - \frac{1}{2}, \quad \mathbf{B}_2(x) = (x - [x])^2 - (x - [x]) + \frac{1}{6},$$

where $[x]$ is the largest integer less than or equal to x . If χ is a primitive Dirichlet character of conductor m , the generalized Bernoulli functions associated to χ are:

$$\mathbf{B}_{1,\chi}(x) = \sum_{a=0}^{m-1} \chi(a) \mathbf{B}_1\left(\frac{x+a}{m}\right), \quad \mathbf{B}_{2,\chi}(x) = m \sum_{a=0}^{m-1} \chi(a) \mathbf{B}_2\left(\frac{x+a}{m}\right).$$

We will write $\mathbf{B}_1(\chi)$ for $\mathbf{B}_{1,\chi}(0)$ and $\mathbf{B}_2(\chi)$ for $\mathbf{B}_{2,\chi}(0)$.

If R is a subring of \mathbf{C} we will write $R[\chi]$ for the ring generated over R by the values of χ . If \mathfrak{X} is a set of Dirichlet characters, then $R[\mathfrak{X}] = R[\chi | \chi \in \mathfrak{X}]$. If M is an R -submodule of \mathbf{C} , then we will write $M[\chi]$ (resp. $M[\mathfrak{X}]$) for $M \cdot R[\chi]$ (resp. $M \cdot R[\mathfrak{X}]$).

The Gauss sum of χ is $\tau(\chi) = \sum_{a=0}^{m-1} \chi(a) e(a/m)$.

1. The cuspidal group. Let $N > 0$ be a positive integer and $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ be one of the groups $\Gamma(N)$, $\Gamma_1(N)$, or $\Gamma_0(N)$. The complete modular curve X_Γ is the union of the affine modular curve $Y_\Gamma = \Gamma \backslash \mathfrak{H}$ and the finite set of cusps, $\mathrm{cusps}(\Gamma) = \Gamma \backslash \mathbb{P}^1(\mathbf{Q})$. The curve X_Γ has a canonical nonsingular projective model defined over \mathbf{Q} [18, Chapter 6]. In this model the set of cusps is invariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Each cusp is rational over $\mathbf{Q}(e(1/N))$, and the ∞ cusp is \mathbf{Q} -rational.

Let $U(\Gamma)$ denote the multiplicative group of modular units over Γ . Thus an element of $U(\Gamma)$ is a modular function g over Γ without zeros or poles on \mathfrak{H} . The group $U(\Gamma)$ may be identified with the group of meromorphic functions on X_Γ whose divisors are supported on the cusps. Let \mathscr{D}_Γ (resp. \mathscr{D}_Γ^0) denote the group of divisors (resp. degree zero divisors) supported on $\mathrm{cusps}(\Gamma)$ and for $g \in U(\Gamma)$ let $\mathrm{div}_\Gamma(g) \in \mathscr{D}_\Gamma^0$ be the divisor of the associated meromorphic function on X_Γ . The cuspidal group on X_Γ is the group

$$C_\Gamma = \mathscr{D}_\Gamma^0 / \mathrm{div}_\Gamma(U(\Gamma)).$$

We will view C_Γ as a subgroup of $\mathrm{Pic}^0(X_\Gamma)$. A theorem of Manin and Drinfeld [2, 11] asserts that C_Γ is a finite group.

Let \mathscr{E}_Γ denote the space of weight two Eisenstein series over Γ . If $E \in \mathscr{E}_\Gamma$, then $E(z) dz$ is a Γ -invariant differential form on \mathfrak{H} . Let $\omega_\Gamma(E)$ denote the 1-form on X_Γ whose pull-back to \mathfrak{H} is $E(z) dz$. Then $\omega_\Gamma(E)$ is regular on Y_Γ but may have simple poles at the cusps. Integration of $\omega_\Gamma(E)$ on Y_Γ induces a homomorphism

$$(1.1) \quad H_1(Y_\Gamma; \mathbf{Z}) \xrightarrow{\int \omega_\Gamma(E)} \mathbf{C}$$

whose image, $\mathscr{P}_\Gamma(E)$, is the group of periods of $\omega_\Gamma(E)$ on Y_Γ . The residual divisor of $\omega_\Gamma(E)$ on X_Γ is the divisor

$$(1.2) \quad \delta_\Gamma(E) = \sum_{x \in \mathrm{cusps}(\Gamma)} r_{\Gamma,E}(x) \cdot (x),$$

where $r_{\Gamma,E}(x) = 2\pi i \cdot \mathrm{Res}_x \omega_\Gamma(E)$. This is a divisor of degree zero with complex coefficients. Let $\mathscr{R}_\Gamma(E)$ be the \mathbf{Z} -submodule of \mathbf{C} generated by the coefficients of $\delta_\Gamma(E)$. Since $r_{\Gamma,E}(x)$ is the integral of $\omega_\Gamma(E)$ over a parabolic cycle about x we have $\mathscr{R}_\Gamma(E) \subseteq \mathscr{P}_\Gamma(E)$.

A simple calculation shows that if $g \in U(\Gamma)$, then $E(z) = (2\pi i)^{-1} g'(z)/g(z)$ is a weight two modular form over Γ . In fact E is an Eisenstein series. This can be seen by verifying directly that E is orthogonal to the space of cusp forms under the Petersson inner product (or see [19, §2.4]). The invariance of $g(z)$ under Γ is equivalent to $\mathscr{P}_\Gamma(E) \subseteq \mathbf{Z}$. If $\mathscr{E}_\Gamma(\mathbf{Z})$ is the lattice of Eisenstein series $E \in \mathscr{E}_\Gamma$ for which $\mathscr{P}_\Gamma(E) \subseteq \mathbf{Z}$, then logarithmic differentiation gives an isomorphism

$$U(\Gamma)/\mathbf{C}^* \xrightarrow{(2\pi i)^{-1} d \log / dz} \mathscr{E}_\Gamma(\mathbf{Z}).$$

Moreover, if $E(z) = (2\pi i)^{-1} g'(z)/g(z)$, then $\delta_\Gamma(E) = \mathrm{div}_\Gamma(g)$. So the cuspidal group can be described using the Eisenstein series:

$$C_\Gamma = \mathscr{D}_\Gamma^0 / \delta_\Gamma(\mathscr{E}_\Gamma(\mathbf{Z})).$$

For an arbitrary \mathbf{Z} -submodule, M , of \mathbf{C} let $\mathcal{E}_\Gamma(M) = \{E \in \mathcal{E}_\Gamma \mid \mathcal{P}_\Gamma(E) \subseteq M\}$. For $E \in \mathcal{E}_\Gamma$ let $A_\Gamma(E) = \mathcal{P}_\Gamma(E)/\mathcal{R}_\Gamma(E)$.

PROPOSITION 1.1. (a) For a \mathbf{Z} -module $M \subseteq \mathbf{C}$ the natural map $\mathcal{E}_\Gamma(\mathbf{Z}) \otimes_{\mathbf{Z}} M \rightarrow \mathcal{E}_\Gamma(M)$ is an isomorphism

(b) For $E \in \mathcal{E}_\Gamma$, $A_\Gamma(E)$ is finite.

PROOF. (a) It suffices to prove this when $M = \mathbf{C}$. By the Manin-Drinfeld theorem the image of $\delta_\Gamma: \mathcal{E}_\Gamma(\mathbf{Z}) \rightarrow \mathcal{D}_\Gamma^0$ has finite index in \mathcal{D}_Γ^0 . Thus extending scalars gives an isomorphism $\delta_\Gamma \otimes 1: \mathcal{E}_\Gamma(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} \rightarrow \mathcal{D}_\Gamma^0 \otimes_{\mathbf{Z}} \mathbf{C}$. The homomorphism $\delta_\Gamma: \mathcal{E}_\Gamma(\mathbf{C}) \rightarrow \mathcal{D}_\Gamma^0 \otimes_{\mathbf{Z}} \mathbf{C}$ is also an isomorphism. Hence the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{E}_\Gamma(\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} & \xrightarrow[\sim]{\delta_\Gamma \otimes 1} & \mathcal{D}_\Gamma^0 \otimes_{\mathbf{Z}} \mathbf{C} \\ \downarrow & \nearrow \delta_\Gamma & \\ \mathcal{E}_\Gamma(\mathbf{C}) & & \end{array}$$

proves (a).

(b) If $0 \neq E_0 \in \mathcal{E}_\Gamma(\mathbf{Z})$, then $(0) \neq \mathcal{R}_\Gamma(E_0) \subseteq \mathbf{Z}$. Hence $\mathbf{Q} \cdot \mathcal{R}_\Gamma(E_0) = \mathbf{Q} \supseteq \mathcal{P}_\Gamma(E_0)$. By (a) it follows that $\mathcal{P}_\Gamma(E) \subseteq \mathbf{Q} \cdot \mathcal{R}_\Gamma(E)$ for any $E \in \mathcal{E}_\Gamma$. Since $\mathcal{R}_\Gamma(E)$, $\mathcal{P}_\Gamma(E)$ are finitely generated, (b) follows. \square

For a prime l not dividing N let T_l be the usual Hecke correspondence on X_Γ and for $a \in (\mathbf{Z}/N\mathbf{Z})^*$ let $\langle a \rangle$ be the associated Nebentypus automorphism of X_Γ [18, 19]. These induce operators on the lattice $\mathcal{E}_\Gamma(\mathbf{Z})$ of integral Eisenstein series and on $\text{Pic}^0(X_\Gamma)$ preserving the cuspidal group C_Γ .

If $\varepsilon_1, \varepsilon_2$ are primitive Dirichet characters defined modulo N and $E \in \mathcal{E}_\Gamma$ satisfies

$$E|T_l = (\varepsilon_1(l) + l\varepsilon_2(l))E$$

for all $l \nmid N$, then we say E has signature $(\varepsilon_1, \varepsilon_2)$. In this case E has Nebentypus character $\varepsilon_1\varepsilon_2$.

Fix an isomorphism

$$\begin{aligned} (\mathbf{Z}/N\mathbf{Z})^* &\rightarrow \text{Gal}(\mathbf{Q}(e(1/N))/\mathbf{Q}) \\ a &\mapsto (\tau_a: e(1/N) \mapsto e(a/N)). \end{aligned}$$

THEOREM 1.2. (a) To every $E \in \mathcal{E}_\Gamma$ there is associated a canonical subgroup $C_\Gamma(E) \subseteq C_\Gamma$ and a canonical perfect pairing

$$C_\Gamma(E) \times A_\Gamma(E) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

(b) If $E \in \mathcal{E}_\Gamma(\mathbf{Q}[\varepsilon_1, \varepsilon_2])$ has signature $(\varepsilon_1, \varepsilon_2)$, then

(i) $\mathcal{P}_\Gamma(E)$, $\mathcal{R}_\Gamma(E)$ are fractional ideals in $\mathbf{Q}[\varepsilon_1, \varepsilon_2]$. Hence $A_\Gamma(E)$ and by duality $C_\Gamma(E)$ inherit natural structures of $\mathbf{Z}[\varepsilon_1, \varepsilon_2]$ -modules.

(ii) $C_\Gamma(E)$ is preserved by the action of $\text{Gal}(\mathbf{Q}(e(1/N))/\mathbf{Q})$, T_l ($l \nmid N$) and $\langle a \rangle$ ($a \in (\mathbf{Z}/N\mathbf{Z})^*$). In terms of the $\mathbf{Z}[\varepsilon_1, \varepsilon_2]$ -module structure:

$$\begin{aligned} \tau_a &\text{ acts as } \varepsilon_1(a); \\ T_l &\text{ acts as } \varepsilon_1(l) + l\varepsilon_2(l); \\ \langle a \rangle &\text{ acts as } \varepsilon_1\varepsilon_2(a). \end{aligned}$$

PROOF. We will define $C_\Gamma(E)$ and prove (a). We refer the reader to [19, §3.2] for the proof of (b).

For $E \in \mathcal{E}_\Gamma$ we have $\delta_\Gamma(E) \in \mathcal{D}_\Gamma^0 \otimes \mathcal{R}_\Gamma(E)$. The map

$$\begin{aligned} \text{Hom}(\mathcal{R}_\Gamma(E); \mathbf{Z}) &\rightarrow \mathcal{D}_\Gamma^0 \\ \phi &\mapsto (1 \otimes \phi)(\delta_\Gamma(E)) \end{aligned}$$

is one-one. Let $\mathcal{D}_\Gamma(E)$ be the image. Then $C_\Gamma(E)$ is the image of $\mathcal{D}_\Gamma(E)$ in C_Γ under the natural map $\mathcal{D}_\Gamma^0 \rightarrow C_\Gamma$.

To see the pairing we observe that because of Proposition 1.1(b), any $\phi \in \text{Hom}(\mathcal{R}_\Gamma(E); \mathbf{Z})$ extends uniquely to a map $\phi: \mathcal{P}_\Gamma(E) \rightarrow \mathbf{Q}$. This gives us a natural pairing

$$\mathcal{D}_\Gamma(E) \times \mathcal{P}_\Gamma(E) \rightarrow \mathbf{Q}.$$

Since $(1 \otimes \phi)(\delta_\Gamma(E)) \in \mathcal{D}_\Gamma(E)$ is principal if and only if $\phi(\mathcal{P}_\Gamma(E)) \subseteq \mathbf{Z}$ this induces a perfect pairing

$$C_\Gamma(E) \times A_\Gamma(E) \rightarrow \mathbf{Q}/\mathbf{Z}. \quad \square$$

To understand the groups $C_\Gamma(E)$ for $E \in \mathcal{E}_\Gamma$ we must therefore have a satisfactory way of understanding the groups $\mathcal{P}_\Gamma(E)$, $\mathcal{R}_\Gamma(E)$. The aim of the next theorem is to describe these groups in terms of special values of L -functions attached to E .

Let $E \in \mathcal{E}_\Gamma$ and $\gamma \in \text{SL}_2(\mathbf{Z})$. Then $E|\gamma$ has a Fourier expansion of the form

$$(1.3) \quad (E|\gamma) = a_0(E|\gamma) + \sum_{n \geq 1} a_n(E|\gamma) e(nz/N)$$

for $z \in \mathfrak{H}$. The constant term $a_0(E|\gamma)$ depends only on the Γ -orbit of $\gamma \cdot i\infty \in \mathbb{P}^1(\mathbf{Q})$. It is related to the L -series of $E|\gamma$ by the formula [19, 2.2.1]

$$(1.4) \quad a_0(E|\gamma) = -L(E|\gamma, 0).$$

Let χ be a primitive Dirichlet character whose conductor m_χ is prime to N . The Dirichlet series

$$(1.5) \quad \bar{\chi}(N) N^s \sum_{n \geq 1} a_n(E) \chi(n) n^{-s}$$

converges absolutely for $\text{Re}(s) > 2$ and extends to a meromorphic function on the complex plane with a possible simple pole at $s = 2$. Let $L(E, \chi, s)$ denote this analytic continuation. If χ is the trivial character we write simply $L(E, s)$. As we will see in the next section (Lemma 2.2), the “special value”

$$(1.6) \quad \Lambda(E, \chi, 1) \stackrel{\text{dfn}}{=} \frac{\tau(\bar{\chi}) L(E, \chi, 1)}{2\pi i}$$

is in $\mathcal{P}_\Gamma(E)[\chi, 1/m_\chi]$ if χ is nontrivial.

If m is an odd prime, let $\chi_m = (\cdot/m)$ be the quadratic character of conductor m . For a primitive nonquadratic Dirichlet character χ whose conductor is a power of m let

$$(1.7) \quad \Lambda_\pm(E, \chi, 1) = \tfrac{1}{2}(\Lambda(E, \chi, 1) \pm \Lambda(E, \chi\chi_m, 1)).$$

In the following S will denote a set of positive primes satisfying

- (1.8) (a) $m \equiv 3 \pmod{4}$, $(m, N) = 1$ for all $m \in S$;
 (b) S intersects every arithmetic progression of the form $\{-1 + Nkr | r \in \mathbf{Z}\}$.

For such a set S , let \mathfrak{X}_S denote the set of primitive Dirichlet characters χ such that $m_\chi \in S$ and χ is not quadratic.

THEOREM 1.3. Let $E \in \mathcal{E}_\Gamma$.

(a) For $x \in \text{cusps}(\Gamma)$

$$r_{\Gamma, E}(x) = e_\Gamma(x) \cdot a_0(E|\gamma_x),$$

where $e_\Gamma(x)$ is the ramification index of x over the j -line $X(1)$, and $\gamma_x \in \text{SL}_2(\mathbf{Z})$ is chosen arbitrarily so that $\gamma_x \cdot i\infty$ is in the Γ -orbit of $\mathbb{P}^1(\mathbf{Q})$ representing x .

(b) Suppose Γ is one of the groups $\Gamma_1(N)$, or $\Gamma(N)$. Let $M \subseteq \mathbf{C}$ be a finitely generated \mathbf{Z} -submodule of \mathbf{C} . Let S be a set of primes satisfying (1.8). Then $E \in \mathcal{E}_\Gamma(M)$ if and only if

- (i) $\mathcal{R}_\Gamma(E) \subseteq M$; and
 (ii) for every $\chi \in \mathfrak{X}_S$

$$\Lambda_\pm(E, \chi, 1) \in M[\chi, 1/m_\chi].$$

PROOF. (a) Let $x \in \text{cusps}(\Gamma)$, $e = e_\Gamma(x)$ and $\gamma = \gamma_x$. The matrices $\alpha = \gamma \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \gamma^{-1}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ generate the parabolic subgroup of Γ stabilizing $\gamma \cdot i\infty$. Thus

$$r_{\Gamma, E}(x) = \int_{z_0}^{\alpha z_0} E(z) dz$$

for every $z_0 \in \mathfrak{R}$. Replacing z_0 by γz_0 we obtain

$$\begin{aligned} r_{\Gamma, E}(x) &= \int_{\gamma z_0}^{\gamma \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \gamma z_0} E(z) dz = \int_{z_0}^{z_0 + e} (E|\gamma)(z) dz \\ &= \frac{1}{N} \int_{z_0}^{z_0 + eN} (E|\gamma)(z) dz \\ &= \frac{1}{N} (eN \cdot a_0(E|\gamma)) \quad (\text{by (1.3)}) \\ &= e_\Gamma(x) \cdot a_0(E|\gamma_x). \end{aligned}$$

The proof of (b) is given in the next section.

2. Special values of L -functions. As in §1, N is a positive integer and Γ is one of the groups, $\Gamma(N)$, $\Gamma_1(N)$ or $\Gamma_0(N)$.

Let χ be a nontrivial primitive Dirichlet character of conductor m with $(m, N) = 1$. If $x, y \in \mathbb{P}^1(\mathbf{Q})$ are Γ -equivalent we will write $\{x, y\}_\Gamma \in H_1(X_\Gamma; \mathbf{Z})$ for the homology class represented by the oriented geodesic in \mathfrak{R} joining x to y . The universal special value $\Lambda(\chi) \in H_1(X_\Gamma; \mathbf{Z}[\chi])$ of the L -function of X_Γ twisted by χ is defined by

$$(2.1) \quad \Lambda(\chi) = \sum_{\substack{a=0 \\ (a, m)=1}}^{m-1} \bar{\chi}(Na) \left\{ x, \frac{Na}{m} \right\}_\Gamma.$$

Here $x \in \mathbb{P}^1(\mathbf{Q})$ may be chosen arbitrarily in the Γ -orbit of N/m . Since the rational numbers Na/m , $(a, m) = 1$, are Γ -equivalent, this expression is well defined. In case $m \equiv \pm 1 \pmod{N}$ we may choose $x = 0$.

In case the conductor of χ is $m = p^n$ for an odd prime $p \equiv 3 \pmod{4}$ and satisfies the congruence $m \equiv \pm 1 \pmod{N}$ we let

$$(2.2) \quad \Lambda_{\pm}(\chi) = \sum_{\substack{a=0 \\ \chi_p(Na) = \pm 1}}^{m-1} \bar{\chi}(Na) \left\{ 0, \frac{Na}{m} \right\}_{\Gamma}.$$

Hence $\Lambda(\chi) = \Lambda_{+}(\chi) + \Lambda_{-}(\chi)$.

If A is a $\mathbf{Z}[\chi]$ -module and $\varphi: H_1(X_{\Gamma}; \mathbf{Z}) \rightarrow A$ is a cohomology class on X_{Γ} , the special value of the L -function of φ twisted by χ is

$$(2.3) \quad \Lambda(\varphi, \chi) = (\varphi \otimes 1)(\Lambda(\chi)).$$

We will also let

$$(2.4) \quad \Lambda_{\pm}(\varphi, \chi) = (\varphi \otimes 1)(\Lambda_{\pm}(\chi)).$$

If S is a set of primes satisfying (1.8) let $\mathbf{Z}[\mathfrak{X}_S]$ denote the ring generated over \mathbf{Z} by the values of the characters $\chi \in \mathfrak{X}_S$. Let \mathfrak{X}_S^{+} (resp. \mathfrak{X}_S^{-}) be the set of $\chi \in \mathfrak{X}_S$ such that $\chi(-1) = 1$ (resp. $\chi(-1) = -1$).

THEOREM 2.1. *Suppose Γ is one of $\Gamma(N)$ or $\Gamma_1(N)$. Let S be a set of primes satisfying (1.8), and A be a $\mathbf{Z}[\mathfrak{X}_S]$ -module. Let $\varphi: H_1(X_{\Gamma}; \mathbf{Z}) \rightarrow A$ be a cohomology class satisfying $\Lambda_{+}(\varphi, \chi) = \Lambda_{-}(\varphi, \chi) = 0$ for every $\chi \in \mathfrak{X}_S^{+}$. Then $\varphi = 0$.*

REMARK. In case the kernel of multiplication by 2 in A is zero, the condition $\Lambda_{+}(\varphi, \chi) = \Lambda_{-}(\varphi, \chi) = 0$ for all $\chi \in \mathfrak{X}_S^{+}$ is equivalent to $\Lambda(\varphi, \chi) = 0$ for all $\chi \in \mathfrak{X}_S$. This follows from the equalities

$$2 \cdot \Lambda_{\pm}(\varphi, \chi) = \Lambda(\varphi, \chi) \pm \Lambda(\varphi, \chi\chi_m)$$

with $\chi \in \mathfrak{X}_S^{+}$ of conductor m .

PROOF OF THEOREM 2.1. Fix a prime ideal $\mathfrak{p} \subseteq \mathbf{Z}[\mathfrak{X}_S]$ and let $A_{\mathfrak{p}}$ be the localization of A at \mathfrak{p} . Let $\varphi_{\mathfrak{p}}$ be the composition of φ with the localization map $A \rightarrow A_{\mathfrak{p}}$.

If $\gamma \in \Gamma$, then the homology class $\{x, \gamma x\}_{\Gamma}$ is independent of $x \in \mathbb{P}^1(\mathbf{Q})$. Thus the map

$$\begin{aligned} \Gamma &\rightarrow H_1(X_{\Gamma}; \mathbf{Z}) \\ \gamma &\mapsto \{x, \gamma x\}_{\Gamma} \end{aligned}$$

is a surjective homomorphism which contains the parabolic elements in its kernel. Define $\Phi_{\mathfrak{p}}: \Gamma \rightarrow A_{\mathfrak{p}}$ by $\Phi_{\mathfrak{p}}(\gamma) = \varphi_{\mathfrak{p}}(\{x, \gamma x\}_{\Gamma})$.

We will prove $\Phi_{\mathfrak{p}} = 0$ in three steps.

(1) If $m \in S$, $m \equiv -1 \pmod{N^2}$ and $\mathfrak{p} \nmid ((m-1)/2)$, then $\varphi_{\mathfrak{p}}(\{0, Na/m\}_{\Gamma}) = 0$ whenever $(m, Na) = 1$.

Let $a \in \mathbf{Z}$ such that $(a, m) = 1$. A standard character calculation shows

$$\begin{aligned} \varphi_p \left(\left\{ 0, \frac{Na}{m} \right\}_\Gamma \right) &= \frac{2}{m-1} \sum_{\chi \neq 1} \chi(Na) \Lambda_{\pm}(\varphi_p, \chi) \\ &\quad + \frac{2}{m-1} \sum_{\substack{k=1 \\ \chi_m(Nk) = \pm 1}}^{m-1} \varphi_p \left(\left\{ 0, \frac{Nk}{m} \right\}_\Gamma \right), \end{aligned}$$

where the sum is over nontrivial characters χ on $(\mathbf{Z}/m\mathbf{Z})^*$ such that $\chi(-1) = 1$ and the \pm sign is chosen according as $\chi_m(Na) = \pm 1$. By the hypotheses of the theorem the terms $\Lambda_{\pm}(\varphi_p, \chi)$ are zero. It follows that $\varphi_p(\{0, Na/m\})$ depends only on the square class of a modulo m . We may therefore assume $a = \pm 1$, the choice of sign being as above.

There is a $k \in \mathbf{Z}$ such that $m = -1 + N^2k$. Let $\beta = \begin{pmatrix} 1 & 0 \\ \pm Nk & 1 \end{pmatrix} \in \Gamma$. Then $\beta \cdot (0) = (0)$ and $\beta \cdot (\mp N) = \pm N/m$. Hence

$$\begin{aligned} \varphi_p \left(\left\{ 0, \frac{Na}{m} \right\}_\Gamma \right) &= \varphi_p \left(\left\{ 0, \frac{\pm N}{m} \right\}_\Gamma \right) = \varphi_p(\{\beta \cdot (0), \beta \cdot (\mp N)\}_\Gamma) \\ &= \varphi_p(\{0, \mp N\}_\Gamma) = \Phi_p \left(\begin{pmatrix} 1 & \mp N \\ 0 & 1 \end{pmatrix} \right) = 0 \end{aligned}$$

since $\begin{pmatrix} 1 & \pm N \\ 0 & 1 \end{pmatrix} \in \Gamma$ is parabolic.

(2) Φ_p vanishes on the subgroup

$$\Gamma'(N) = \left\{ \begin{pmatrix} a & Nb \\ Nc & d \end{pmatrix} \in \Gamma(N) \mid a \equiv d \equiv \pm 1 \pmod{N^2} \right\}$$

of Γ .

Let $\gamma = \begin{pmatrix} a & Nb \\ Nc & d \end{pmatrix} \in \Gamma'(N)$. Without loss of generality we may assume that $d \equiv -1 \pmod{N^2}$. We may choose $k \in \mathbf{Z}$ so that $m = d + N^2bk$ is a prime in S satisfying the congruence $m \equiv -1 \pmod{4N^2p}$. Then $(m-1)/2 = (m+1)/2 - 1 \equiv -1 \pmod{p}$ so $p \nmid ((m-1)/2)$. Thus m satisfies the hypotheses of (1).

Let $\beta = \begin{pmatrix} 1 & 0 \\ Nk & 1 \end{pmatrix} \in \Gamma$. Then

$$\begin{aligned} \Phi_p(\gamma) &= \varphi_p \left(\left\{ 0, \frac{Nb}{d} \right\}_\Gamma \right) = \varphi_p \left(\left\{ \beta \cdot (0), \beta \cdot \left(\frac{Nb}{d} \right) \right\}_\Gamma \right) \\ &= \varphi_p \left(\left\{ 0, \frac{Nb}{m} \right\}_\Gamma \right) = 0; \end{aligned}$$

the last equality being a consequence of (1).

(3) $\Phi_p = 0$.

LEMMA (FRICKE, WOHLFAHRT). *Let N' be a positive integer divisible by N . Then $\Gamma(N)$ is generated by $\Gamma(N')$ and the parabolic elements of $\Gamma(N)$. \square*

As a consequence we see that Γ is generated by $\Gamma'(N)$ and the parabolic elements of Γ . If $\Gamma = \Gamma(N)$, then this follows from the lemma with $N' = N^2$. Since $\Gamma_1(N)$ is generated by $\Gamma(N)$ and the parabolic element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the statement is valid for $\Gamma = \Gamma_1(N)$ as well.

Since $\Phi_{\mathfrak{p}}$ vanishes on $\Gamma'(N)$ and on parabolic elements of Γ , (3) follows.

The homomorphism $\Gamma \rightarrow H_1(X_{\Gamma}; \mathbf{Z})$ is surjective. Therefore $\varphi_{\mathfrak{p}} = 0$. Since this is true for all \mathfrak{p} we have $\varphi = 0$. \square

We now return to the proof of Theorem 1.3(b). Let $E \in \mathcal{E}_{\Gamma}$ and let $\mathcal{R} = \mathcal{R}_{\Gamma}(E)$, $\mathcal{P} = \mathcal{P}_{\Gamma}(E)$. Integration of $\omega_{\Gamma}(E)$ on Y_{Γ} induces a map

$$H_1(Y_{\Gamma}; \mathbf{Z}) \rightarrow \mathcal{P}/\mathcal{R} = A_{\Gamma}(E).$$

Since the parabolic classes are contained in the kernel, this map factors through a homomorphism

$$(2.5) \quad \varphi_E: H_1(X_{\Gamma}; \mathbf{Z}) \rightarrow A_{\Gamma}(E).$$

We will refer to this as the parabolic cohomology class associated to E .

If χ is a Dirichlet character of conductor m , let $\varphi_{E,\chi}$ be the composition

$$(2.6) \quad \varphi_{E,\chi}: H_1(X_{\Gamma}; \mathbf{Z}) \rightarrow A_{\Gamma}(E) \rightarrow \mathcal{P}[\chi, 1/m]/\mathcal{R}[\chi, 1/m].$$

LEMMA 2.2. *Let $p \equiv 3 \pmod{4}$ be an odd prime and $\chi \neq 1$ be an even primitive Dirichlet character of conductor $m = p^n$. Suppose $m \equiv \pm 1 \pmod{N}$. Then*

- (1) $\Lambda_{\pm}(E, \chi, 1) \in \mathcal{P}[\chi, 1/m]$; and
- (2) $\Lambda_{\pm}(\varphi_{E,\chi}, \chi) \equiv \Lambda_{\pm}(E, \chi, 1) \pmod{\mathcal{R}[\chi, 1/m]}$.

PROOF. We will use Schoeneberg's cocycle $\Pi_E: \mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathbf{C}$ [17, 19]. The basic properties are:

- (a) $\Pi_E(\alpha\beta) = \Pi_E(\alpha) + \Pi_{E|\alpha}(\beta)$ for $\alpha, \beta \in \mathrm{SL}_2(\mathbf{Z})$.
- (b) If $\gamma \in \Gamma$, then

$$\Pi_E(\gamma) = \int_{z_0}^{\gamma z_0} E(z) dz$$

for arbitrary $z_0 \in \mathfrak{H}$.

- (c) If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ and $c \neq 0$, then

$$\Pi_E(\alpha) = \frac{a}{c}a_0(E) + \frac{d}{c}a_0(E|\alpha) + D_E\left(\frac{a}{c}\right),$$

where

$$D_E\left(\frac{a}{c}\right) = \frac{1}{2\pi i} L\left(E\left(\begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix}, 1\right)\right).$$

The term $D_E(a/c)$ is usually expressed in terms of generalized Dedekind sums but we will not need these expressions here. The formula

$$(2.7) \quad \Lambda(E, \chi, 1) = \sum_{a=1}^{m-1} \bar{\chi}(Na) D_E\left(\frac{Na}{m}\right)$$

is an easy consequence of (c) (see [19, 3.1.1]).

Because of (b) we have

$$\varphi_E(\{x, \gamma x\}_{\Gamma}) \equiv \Pi_E(\gamma) \pmod{\mathcal{R}}$$

for every $x \in \mathbb{P}^1(\mathbf{Q})$ and $\gamma \in \Gamma$. If $a \in \mathbf{Z}$ and $(a, m) = 1$, let $\gamma_a \in \Gamma$ be any element for which $\gamma_a \cdot (0) = (Na/m)$. So $\gamma_a = \begin{pmatrix} b & Na \\ Na' & m \end{pmatrix}$ for some choice of $a', b \in \mathbf{Z}$. Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\gamma_a w = \begin{pmatrix} Na & -b \\ m & -Na' \end{pmatrix}$ and by (c)

$$\Pi_E(\gamma_a w) = \frac{Na}{m} a_0(E) - \frac{Na'}{m} a_0(E|w) + D_E\left(\frac{Na}{m}\right).$$

Since $N \cdot a_0(E), Na_0(E|w) \in \mathcal{R}$ by Theorem 1.3(a), we see

$$\Pi_E(\gamma_a w) \equiv D_E\left(\frac{Na}{m}\right) \pmod{\mathcal{R}\left[\frac{1}{m}\right]}.$$

On the other hand, by (a), (c) we have

$$\Pi_E(\gamma_a w) = \Pi_E(\gamma_a) + \Pi_E(w), \quad \Pi_E(w) = D_E(0).$$

We therefore have the congruence

$$\Pi_E(\gamma_a) \equiv D_E\left(\frac{Na}{m}\right) - D_E(0) \pmod{\mathcal{R}\left[\frac{1}{m}\right]}.$$

Calculating modulo $\mathcal{R}[\chi, 1/m]$ we find

$$\begin{aligned} \Lambda_{\pm}(\varphi_{E,\chi}, \chi) &= \sum_{\substack{a=0 \\ \chi_p(Na) = \pm 1}}^{m-1} \bar{\chi}(Na) \varphi_{E,\chi}\left(\left\{0, \frac{Na}{m}\right\}_{\Gamma}\right) \equiv \sum_{\chi_p(Na) = \pm 1} \bar{\chi}(Na) \Pi_E(\gamma_a) \\ &\equiv \sum_{\chi_p(Na) = \pm 1} \bar{\chi}(Na) D_E\left(\frac{Na}{m}\right) \quad (\text{since } \chi \text{ is nontrivial}) \\ &= \frac{1}{2} \sum_{\substack{a=0 \\ (a,m)=1}}^{m-1} \bar{\chi}(Na) (1 \pm \chi_p(Na)) D_E\left(\frac{Na}{m}\right) = \Lambda_{\pm}(E, \chi, 1). \end{aligned}$$

This proves both (1) and (2) of Lemma 2.2. \square

PROOF OF THEOREM 1.3(b). There is a number field F such that for any other number field K linearly disjoint from F the natural map $M \otimes K \rightarrow MK$ is an isomorphism. Let d_F be the discriminant of F and

$$S' = \{m \in S \mid (d_F, (m-1)/2) = 1\}.$$

Then S' satisfies (1.8) and if $\chi \in \mathfrak{X}_{S'}$, then $\mathbf{Q}[\chi]$ is linearly disjoint from F .

Let p be prime and $M_p \subseteq \mathbf{C}$ be the localization of M at p . Let $A_p = \mathbf{C}/M_p[\mathfrak{X}_{S'}]$. Since $\mathcal{R}_{\Gamma}(E) \subseteq M$, integration of $\omega(E)$ induces a parabolic cohomology class $\varphi_p: H_1(X_{\Gamma}; \mathbf{Z}) \rightarrow A_p$. By hypothesis $\Lambda_{\pm}(E, \chi, 1) \in M_p[\mathfrak{X}_{S'}]$ for every $\chi \in \mathfrak{X}_{S'}^+$ with $m_{\chi} \neq p$. By Lemma 2.2 we have $\Lambda_{\pm}(\varphi_p, \chi) = 0$ for these χ . Then by Theorem 2.1, $\varphi_p = 0$. Hence $\mathcal{P}_{\Gamma}(E) \subseteq M_p[\mathfrak{X}_{S'}]$. Since this is true for every prime p we have $\mathcal{P}_{\Gamma}(E) \subseteq M[\mathfrak{X}_{S'}]$. On the other hand because of Proposition 1.1(b), $\mathcal{P}_{\Gamma}(E) \subseteq \mathbf{Q} \cdot \mathcal{R}_{\Gamma}(E) \subseteq \mathbf{Q} \cdot M$. Thus

$$\mathcal{P}_{\Gamma}(E) \subseteq \mathbf{Q} \cdot M \cap M[\mathfrak{X}_{S'}].$$

Since $\mathbf{Q}[\mathfrak{X}_{S'}] \cap F = \mathbf{Q}$, we have $\mathbf{Q} \cdot M \cap M[\mathfrak{X}_{S'}] = M$. Therefore $E \in \mathcal{E}_{\Gamma}(M)$ as claimed. \square

3. Modular units. Let $U = \bigcup_{N>0} U(\Gamma(N))$ be the group of modular units of all levels. For each positive integer N let U_N be the subgroup of U consisting of the units $g \in U$ such that $g^m \in U(\Gamma(N))$ for some m .

A theorem of Kubert [7, 8] gives a complete description of U_N/\mathbf{C}^* by describing both its elements and its structure as an $\mathrm{SL}_2(\mathbf{Z})$ -module. In this section we recall Kubert's theorem and use Theorem 1.3 to give another proof.

For $N > 0$ let D_N be the free abelian group generated by the symbols

$$\mathbf{x} \in \frac{1}{N} \mathbf{Z}^2 / \mathbf{Z}^2 \setminus \{\mathbf{0}\}.$$

Let $D = \mathrm{inj} \lim D_N$. Let $R \subseteq D$ be the subgroup of “distribution relations” generated by the elements

$$(3.1) \quad r_n(\mathbf{x}) = \begin{cases} \sum_{n\mathbf{y}=\mathbf{x}} (\mathbf{y}) - (\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{0}, \\ \sum_{\substack{n\mathbf{y}=(\mathbf{0}) \\ \mathbf{y} \neq \mathbf{0}}} (\mathbf{y}) & \text{if } \mathbf{x} = \mathbf{0}, \end{cases}$$

where n is a positive integer and $\mathbf{x} \in \mathbf{Q}^2/\mathbf{Z}^2$. Let $\mathbb{U} = D/R$. The universal (punctured) distribution is the natural map

$$(3.2) \quad \mathbf{Q}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\} \rightarrow \mathbb{U}.$$

For a positive integer N let $\mathbb{U}_N \subseteq \mathbb{U}$ be the image of D_N .

The natural right action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{Q}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$ defined by

$$(3.3) \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \mathbf{x} = (x_1, x_2) \mapsto \mathbf{x}\gamma = (ax_1 + cx_2, bx_1 + dx_2)$$

extends to an action on D leaving R invariant. The induced action of $\mathrm{SL}_2(\mathbf{Z})$ on \mathbb{U} preserves the subgroup \mathbb{U}_N . For $u \in \mathbb{U}$ and $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ we will write γ^*u for the result of applying γ to u . Let \mathbb{U}^+ be the subgroup of even elements of \mathbb{U} , that is, the elements $u \in \mathbb{U}$ for which $(-1)^*u = u$. Since (-1) is in the center of $\mathrm{SL}_2(\mathbf{Z})$, \mathbb{U}^+ is an $\mathrm{SL}_2(\mathbf{Z})$ -submodule of \mathbb{U} . Let $\mathbb{U}_N^+ = \mathbb{U}_N \cap \mathbb{U}^+$.

For $\mathbf{x} = (x_1, x_2) \in \mathbf{Q}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$ define the function $j_{\mathbf{x}}(z)$ on \mathbb{R} by its Fourier expansion:

$$(3.4) \quad j_{\mathbf{x}}(z) = e\left(\frac{1}{4}\mathbf{B}_2(x_1)z\right) \prod_{\substack{k \equiv x_1 \pmod{\mathbf{Z}} \\ k > 0}} (1 - e(kz + x_2)).$$

Then $j_{\mathbf{x}}(z)$ has no poles or zeros on \mathbb{R} . Let J_N be the multiplicative group generated by \mathbf{C}^* and the functions $j_{\mathbf{x}}(z)$, $\mathbf{x} \in \frac{1}{N} \mathbf{Z}^2 / \mathbf{Z}^2 \setminus \{\mathbf{0}\}$, and let $J = \bigcup_{N>0} J_N$. It is not difficult to verify the distribution relations:

$$(3.5) \quad \prod_{n\mathbf{y}=\mathbf{x}} j_{\mathbf{y}} = j_{\mathbf{x}}, \quad \prod_{\substack{n\mathbf{y}=\mathbf{0} \\ \mathbf{y} \neq \mathbf{0}}} j_{\mathbf{y}} = 1$$

for $n > 0$ and $\mathbf{x} \in \mathbf{Q}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$. Hence the homomorphism $D \rightarrow J$ defined by $\mathbf{x} \mapsto j_{\mathbf{x}}(z)$ factors through the universal distribution to give a homomorphism $j: \mathbb{U} \rightarrow J$. Let J_N^+ be the subgroup of J_N generated by \mathbf{C}^* and $j(\mathbb{U}_N^+)$.

THEOREM (KUBERT). (1) $U_N = J_N^+$.

(2) The homomorphism $\mathbb{U}_N^+ \rightarrow U_N/\mathbf{C}^*$, induced by j , is an isomorphism of $\mathrm{SL}_2(\mathbf{Z})$ -modules.

We will restrict our attention to what appears to be the most difficult part of the proof, namely the inclusion $J_N^+ \subseteq U_N$. For the rest of the proof we refer the reader to [7 or 6, Chapter 4].

We will prove $J_N^+ \subseteq U_N$ by verifying the corresponding statement about weight two Eisenstein series.

Let $\mathcal{E}(\mathbf{Z}) = \bigcup_{N>0} \mathcal{E}_{\Gamma(N)}(\mathbf{Z})$. Logarithmic differentiation gives an isomorphism

$$(3.6) \quad U/\mathbf{C}^* \xrightarrow[\sim]{(2\pi i)^{-1}d\log/dz} \mathcal{E}(\mathbf{Z}).$$

The image of U_N/\mathbf{C}^* is $\mathcal{E}(\mathbf{Z}) \cap \mathcal{E}_{\Gamma(N)}$.

For $\mathbf{x} = (x_1, x_2) \in \mathbf{Q}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$, let $l_{\mathbf{x}}$ be the function on \mathfrak{H} defined by

$$(3.7) \quad \begin{aligned} l_{\mathbf{x}}(z) &= \frac{1}{2\pi i} j'_{\mathbf{x}}(z)/j_{\mathbf{x}}(z) \\ &= \frac{1}{4} \mathbf{B}_2(x_1) - \sum_{\substack{k \equiv x_1 \pmod{\mathbf{Z}} \\ k \in \mathbf{Q}^+}} k \cdot \sum_{m=1}^{\infty} e(m(kz + x_2)). \end{aligned}$$

Let L_N be the additive group generated by the functions $l_{\mathbf{x}}$, $\mathbf{x} \in \frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$. Define $l: \mathbb{U}_N \rightarrow L_N$ to be the composition

$$\mathbb{U}_N \xrightarrow{j} J_N \xrightarrow{(2\pi i)^{-1}d\log/dz} L_N$$

and let $L_N^+ = l(\mathbb{U}_N^+)$.

The inclusion $J_N^+ \subseteq U_N$ is an immediate consequence of the following proposition. Moreover, we see that the functions in J_N^+ are modular of level $N' = 4 \cdot \mathrm{lcm}(6, N^2)$.

PROPOSITION 3.1. (1) $L_N^+ \subseteq \mathcal{E}_{\Gamma(N)}$;

(2) $L_N^+ \subseteq \mathcal{E}_{\Gamma(N')}(\mathbf{Z})$, where $N' = 4 \cdot \mathrm{lcm}(6, N^2)$.

PROOF. (1) For $\mathbf{x} \in \frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$ let

$$(3.8) \quad \phi_{\mathbf{x}} = l_{\mathbf{x}} + l_{-\mathbf{x}}, \quad \eta_{\mathbf{x}} = j_{\mathbf{x}} \cdot j_{-\mathbf{x}}.$$

A comparison of Fourier expansions shows that $\eta_{\mathbf{x}}$ is a nonzero multiple of the Siegel unit $g_{\mathbf{x}}$ [6, p. 29]. Hence for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$,

$$\eta_{\mathbf{x}}(\gamma z) = c \cdot \eta_{\mathbf{x}\gamma}(z)$$

for a root of unity c . Taking logarithmic derivatives we find

$$(3.9) \quad \phi_{\mathbf{x}}|_{\gamma} = \phi_{\mathbf{x}\gamma}.$$

In case $\gamma \in \Gamma(N)$ this equality says $\phi_{\mathbf{x}}|_{\gamma} = \phi_{\mathbf{x}}$. Thus $\phi_{\mathbf{x}} \in \mathcal{E}_{\Gamma(N)}$.

For $E \in L_N^+$ there is a $u \in \mathbb{U}_N^+$ such that $l(u) = E$. Then $2 \cdot E = l(u + (-1)^*u)$ is a \mathbf{Z} -linear combination of the functions $\phi_{\mathbf{x}}$, $\mathbf{x} \in \frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$. In particular, $E \in \mathcal{E}_{\Gamma(N)}$.

(2) Let $\Gamma = \Gamma(N')$, $N' = 4 \cdot \mathrm{lcm}(6, N^2)$. Let $E \in L_N^+$. By (1) we have $E \in \mathcal{E}_{\Gamma}$. We will use Theorem 1.3 to show $E \in \mathcal{E}_{\Gamma}(\mathbf{Z})$.

We begin by proving $\mathcal{R}_\Gamma(E) \subseteq \mathbf{Z}$. By the argument of (1) we know that $2E$ is a \mathbf{Z} -linear combination of the Eisenstein series ϕ_x , $x \in \frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$. Hence it suffices to prove $\mathcal{R}_\Gamma(\phi_x) \subseteq 2\mathbf{Z}$ for each x . Since the ramification index of every cusp on X_Γ over $X(1)$ is N' , Theorem 1.3(a) shows that $\mathcal{R}_\Gamma(\phi_x)$ is generated as a \mathbf{Z} -module by the numbers

$$N' \cdot a_0(\phi_{x_\gamma}), \quad \gamma \in \mathrm{SL}_2(\mathbf{Z}).$$

If $x_\gamma = (x_1, x_2)$, then an examination of the Fourier expansion of ϕ_{x_γ} reveals $a_0(\phi_{x_\gamma}) = \frac{1}{2}\mathbf{B}_2(x_1)$. Hence $N' \cdot a_0(\phi_{x_\gamma}) \in 2\mathbf{Z}$, as desired.

To complete the proof we must study the values $\Lambda(E, \chi, 1)$ for $E \in L_N^+$ and nontrivial primitive Dirichlet characters χ of conductor prime to N .

We may extend the definition of $L(E, \chi, s)$ to all $E \in L_N$ by setting

$$L(E, \chi, s) = \bar{\chi}(N) N^s \sum_{n=1}^{\infty} a_n(E) \chi(n) n^{-s}, \quad \mathrm{Re}(s) > 2,$$

if $E = a_0(E) + \sum_{n=1}^{\infty} a_n(E) e(nz/N)$. For $x = (x_1, x_2) \in \frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$ it is not hard to check the equality

$$(3.10) \quad L(l_x, \chi, s) = -\zeta(s-1, \chi, x_1) \cdot Z(s, \chi, x_2),$$

where

$$(3.11) \quad \begin{aligned} \zeta(s, \chi, x_1) &= \bar{\chi}(N) \sum_{\substack{k \equiv x_1 \pmod{\mathbf{Z}} \\ k > 0}} \chi(Nk) k^{-s}, \\ Z(s, \chi, x_2) &= \sum_{n=1}^{\infty} \chi(n) e(nx_2) n^{-s}. \end{aligned}$$

Since ζ and Z are Hurwitz zeta functions they converge absolutely in the half-plane $\mathrm{Re}(s) > 1$ and extend to meromorphic functions with possible simple poles only at $s = 1$. For a general Hurwitz zeta function $H(s)$ represented by a Dirichlet series $\sum_{n=1}^{\infty} \psi(n) n^{-s}$ with a periodic function ψ of period A , the residue at $s = 1$ is given by

$$\frac{1}{A} \sum_{n=1}^A \psi(n).$$

Hence $\zeta(s, \chi, x_1)$ and $Z(s, \chi, x_2)$ are regular at $s = 1$. Since any $E \in L_N$ is a linear combination of functions l_x we have the following

LEMMA 3.2. *If $E \in L_N$ and χ is a nontrivial primitive Dirichlet character, then the Dirichlet series $L(E, \chi, s)$ converges absolutely for $\mathrm{Re}(s) > 2$ and extends to an entire function of $s \in \mathbf{C}$. \square*

For $E \in L_N$ we may define

$$\Lambda(E, \chi, 1) = \frac{\tau(\bar{\chi}) L(E, \chi, 1)}{2\pi i}$$

as we did for Eisenstein series E .

Let

$$(3.12) \quad S = \{m \text{ prime} \mid m \equiv 3 \pmod{4} \text{ and } m \equiv -1 \pmod{N}\}.$$

Clearly, S satisfies (1.8). For $\chi \in \mathfrak{X}_S^+$ of conductor m and $E \in L_N$ let

$$\Lambda_{\pm}(E, \chi, 1) = \frac{1}{2}(\Lambda(E, \chi, 1) \pm \Lambda(E, \chi\chi_m, 1)).$$

For the rest of the proof we will fix a prime $m \in S$ and a character $\chi \in \mathfrak{X}_S^+$ of conductor m . We will show $\Lambda_{\pm}(E, \chi, 1) \in \mathbf{Z}[\chi, 1/m]$ for $E \in L_N^+$.

Let $K = \mathbf{Q}(e(1/Nm))$ and fix an isomorphism

$$\begin{aligned} (\mathbf{Z}/n\mathbf{Z})^* &\rightarrow G = \text{Gal}(K/\mathbf{Q}(e(1/N))) \\ r &\mapsto (\tau_r: e(1/m) \mapsto e(r/m)). \end{aligned}$$

For a $\mathbf{Z}[G]$ -module V let $V(\chi)$ be the $\mathbf{Z}[\chi, 1/m]$ -submodule of $V \otimes_{\mathbf{Z}} \mathbf{Z}[\chi, 1/m]$ on which the *squares* in G act via χ . Hence

$$V(\chi) = \left\{ v \in V \otimes \mathbf{Z}[\chi, 1/m] \mid \tau_r(v) = \chi(r)v \text{ for every quadratic residue } r \in (\mathbf{Z}/m\mathbf{Z})^* \right\}.$$

Let W be the group of units in K written additively. Thus W is generated by the symbols $\{\omega\}$, $\omega \in \mathfrak{o}_K^*$, subject to the relations $\{\omega_1\omega_2\} = \{\omega_1\} + \{\omega_2\}$ for $\omega_1, \omega_2 \in \mathfrak{o}_K^*$. Of course W is a $\mathbf{Z}[G]$ -module.

Define

$$\lambda: W \otimes_{\mathbf{Z}} \mathbf{Z}[\chi, 1/m] \rightarrow \mathbf{C}/\mathbf{Z}[\chi, 1/m]$$

by

$$\lambda(\{\omega\} \otimes \alpha) = \alpha \cdot \frac{1}{2\pi i} \log(\omega) \pmod{\mathbf{Z}[\chi, 1/m]}.$$

Let

$$\pi: \mathbf{C} \rightarrow \mathbf{C}/\mathbf{Z}[\chi, 1/m]$$

be the natural projection. If $w = \{\omega\} \otimes \alpha$ is an element of $W \otimes \mathbf{Z}[\chi, 1/m]$, we write \bar{w} for $\{\bar{\omega}\} \otimes \alpha$ where $\bar{\omega}$ is the complex conjugate of ω .

We need two lemmas. The first is concerned with special values of L -functions. The second is about units in cyclotomic fields.

LEMMA 3.3. *For either choice of sign \pm , there is a distribution $w_{\chi}^{\pm}: \mathbb{U}_N \rightarrow W(\chi)$ such that*

(a) *The diagram*

$$\begin{array}{ccccc} \mathbb{U}_N & \xrightarrow{w_{\chi}^{\pm}} & & & W(\chi) \\ \downarrow \iota & & & & \downarrow \lambda \\ L_N & \xrightarrow{\Lambda_{\pm}(\cdot, \chi, 1)} & \mathbf{C} & \xrightarrow{\pi} & \mathbf{C}/\mathbf{Z}[\chi, 1/m] \end{array}$$

is commutative;

(b) *For $u \in \mathbb{U}_N$, $w_{\chi}^{\pm}((-1)^*u) = \overline{-w_{\chi}^{\pm}(u)}$.*

LEMMA 3.4. *If $w \in W(\chi)$ and $w + \bar{w} = 0$, then $w = 0$.*

Before proving Lemmas 3.3 and 3.4 we show how they can be used to complete the proof of the proposition. Let $E \in L_N^+$ and $u \in \mathbb{U}_N^+$ such that $E = \ell(u)$. Set

$w = w_{\chi}^{\pm}(u)$. Since $(-1)^*u = u$ we have $w + \bar{w} = 0$ by Lemma 3.3(b). Lemma 3.4 then tells us $w = 0$. By Lemma 3.3(a)

$$\pi(\Lambda_{\pm}(E, \chi, 1)) = \lambda(w) = 0.$$

Therefore $\Lambda_{\pm}(E, \chi, 1) \in \mathbf{Z}[\chi, 1/m]$. By Theorem 1.3(b), $E \in \mathcal{E}_{\Gamma}(\mathbf{Z})$ as was to be shown.

PROOF OF LEMMA 3.3. To define w_{χ}^{\pm} we need the generalized Bernoulli functions

$$(3.13) \quad \mathbf{B}_{1,\psi}^{\pm}(x) = \frac{1}{2}(\mathbf{B}_{1,\psi}(x) \pm \mathbf{B}_{1,\psi\epsilon}(x)),$$

where ψ is a primitive nonquadratic Dirichlet character of conductor m , $x \in \mathbf{Q}$ and $\epsilon = \chi_m$. Also, let $\mathbf{B}_1^{\pm}(\psi) = \mathbf{B}_{1,\psi}^{\pm}(0)$. The basic properties we will use are:

$$(1) \quad \mathbf{B}_1^{\pm}(\psi) = \sum_{\substack{a=1 \\ \epsilon(a)=\pm 1}}^{m-1} \psi(a) \cdot a/m;$$

$$(2) \quad \mathbf{B}_{1,\psi}^{\pm}(x) = \mathbf{B}_1^{\pm}(\psi) - \psi(-1) \cdot \sum_{\substack{a=0 \\ \epsilon(a)=\mp 1}}^x \psi(a),$$

where Σ^* means the terms corresponding to the endpoints $a = 0, x$ are weighted by a factor of $\frac{1}{2}$,

$$(3) \quad \mathbf{B}_{1,\psi}^{\pm}(x) = -\psi(-1) \cdot \mathbf{B}_{1,\psi}^{\mp}(-x).$$

(1) and (3) are straightforward calculations and (2) is a consequence of a corresponding formula for $\mathbf{B}_{1,\psi}(x)$ (see [19, 3.6.2]). Note that if $x \in \frac{m}{N}\mathbf{Z}$, then Σ^* may be replaced by Σ . Thus (1) and (2) show $\mathbf{B}_{1,\psi}^{\pm}(x) \in \mathbf{Z}[\chi, 1/m]$.

Let $\mathbf{x} \in \frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$. Since $(m, N) = 1$ we may represent \mathbf{x} by an element $(x_1, x_2) \in \frac{1}{N}\mathbf{Z}^2$ with $x_1 \in \frac{m}{N}\mathbf{Z}$. For our fixed character χ define $w^{\pm}(\mathbf{x}, \chi) \in W \otimes \mathbf{Z}[\chi, 1/m]$ by the formula

$$w^{\pm}(\mathbf{x}, \chi) = \sum_{a=1}^{m-1} \left\{ \frac{1 - e(x_2 + a/m)}{1 - e(x_2 + 1/m)} \right\} \otimes \bar{\chi}(a) \mathbf{B}_{1,\chi}^{\pm \epsilon(a)}(x_1).$$

For a quadratic residue $r \in (\mathbf{Z}/n\mathbf{Z})^*$ an easy calculation shows

$$\tau_r(w^{\pm}(\mathbf{x}, \chi)) = \chi(r) w^{\pm}(\mathbf{x}, \chi).$$

Hence $w^{\pm}(\mathbf{x}, \chi) \in W(\chi)$.

The formulas

$$\prod_{r=0}^{n-1} \left(1 - e\left(\frac{x+r}{n} + \frac{a}{m}\right) \right) = \left(1 - e\left(x + \frac{na}{m}\right) \right)$$

and

$$\sum_{s=0}^{n-1} \mathbf{B}_{1,\chi}^{\pm}\left(\frac{x+s}{n}\right) = \bar{\chi}(n) \cdot \mathbf{B}_{1,\chi}^{\pm}(x)$$

for $(n, m) = 1$ show that $w^{\pm}(\cdot, \chi)$ is a distribution on $\frac{1}{N}\mathbf{Z}^2/\mathbf{Z}^2 \setminus \{\mathbf{0}\}$. Hence $w^{\pm}(\cdot, \chi)$ induces a homomorphism

$$w_{\chi}^{\pm}: \mathbb{U}_N \rightarrow W(\chi).$$

(a) For $x \in \mathbf{Q}$ let

$$\zeta(s, x) = \sum_{\substack{k \equiv x \pmod{\mathbf{Z}} \\ k > 0}} k^{-s} \quad \text{and} \quad Z(s, x) = \sum_{n=1}^{\infty} e(nx) n^{-s}.$$

These are Hurwitz zeta functions and thus have analytic continuations with possible simple poles only at $s = 1$. In fact if x is not an integer, then $Z(s, x)$ is regular at $s = 1$. Moreover we have the formulas

$$\zeta(0, x) = -\mathbf{B}_1(x), \quad Z(1, x) = -\log(1 - e(x))$$

for $x \in \mathbf{Q} \setminus \mathbf{Z}$ [21, p. 271]. Here \log denotes the principal branch of the logarithm.

For a primitive character ψ of conductor m , the functions $\zeta(s, \psi, x_1)$ and $Z(s, \psi, x_2)$ can be expressed in terms of the functions $\zeta(s, \cdot)$ and $Z(s, \cdot)$ by

$$\zeta(s, \psi, x_1) = m^{-s} \sum_{a=1}^{m-1} \psi(a) \zeta\left(s, \frac{x_1 + a}{m}\right)$$

(recall $x_1 \in m\mathbf{Z}/N$) and

$$\tau(\bar{\psi}) Z(s, \psi, x_2) = \sum_{a=1}^{m-1} \bar{\psi}(a) Z\left(s, x_2 + \frac{a}{m}\right).$$

Thus

$$\begin{aligned} \zeta(0, \psi, x_1) &= -\mathbf{B}_{1,\psi}(x_1), \\ \tau(\bar{\psi}) Z(1, \psi, x_2) &= - \sum_{a=1}^{m-1} \bar{\psi}(a) \log\left(\frac{1 - e(x_2 + a/m)}{1 - e(x_2 + 1/m)}\right). \end{aligned}$$

Hence

$$\begin{aligned} \Lambda(l_x, \psi, 1) &= -\zeta(0, \psi, x_1) \cdot \frac{\tau(\bar{\psi})}{2\pi i} Z(1, \psi, x_2) \\ &= -\mathbf{B}_{1,\psi}(x_1) \cdot \sum_{a=1}^{m-1} \bar{\psi}(a) \cdot \frac{1}{2\pi i} \log\left(\frac{1 - e(x_2 + a/m)}{1 - e(x_2 + 1/m)}\right). \end{aligned}$$

Thus for our fixed character χ we have

$$\begin{aligned} \Lambda_{\pm}(l_x, \chi, 1) &= - \sum_{a=1}^{m-1} \bar{\chi}(a) \cdot \frac{1}{2} (\mathbf{B}_{1,\chi}(x_1) \pm \epsilon(a) \mathbf{B}_{1,\chi\epsilon}(x_1)) \\ &\quad \cdot \frac{1}{2\pi i} \log\left(\frac{1 - e(x_2 + a/m)}{1 - e(x_2 + 1/m)}\right) \\ &= - \sum_{a=1}^{m-1} \bar{\chi}(a) \mathbf{B}_{1,\chi}^{\pm\epsilon(a)}(x_1) \frac{1}{2\pi i} \log\left(\frac{1 - e(x_2 + a/m)}{1 - e(x_2 + 1/m)}\right). \end{aligned}$$

The commutativity of the diagram in (a) is now clear.

(b) We use property (3) of the functions $\mathbf{B}_{1,\chi}^\pm$ to compute

$$\begin{aligned} w^\pm(-\mathbf{x}, \chi) &= \sum_{a=1}^{m-1} \left\{ \frac{1 - e(-x_2 + a/m)}{1 - e(x_2 - 1/m)} \right\} \otimes \bar{\chi}(a) \mathbf{B}_{1,\chi}^{\pm \epsilon(a)}(-x_1) \\ &= \sum_{a=1}^{m-1} \left\{ \frac{1 - e(-x_2 - a/m)}{1 - e(-x_2 - 1/m)} \right\} \otimes \bar{\chi}(-a) \mathbf{B}_{1,\chi}^{\pm \epsilon(-a)}(-x_1) \\ &= - \sum_{a=1}^{m-1} \left\{ \frac{1 - e(-x_2 - a/m)}{1 - e(-x_2 - 1/m)} \right\} \otimes \bar{\chi}(a) \mathbf{B}_{1,\chi}^{\pm \epsilon(a)}(x_1) \\ &= - \overline{w^\pm(\mathbf{x}, \chi)}. \end{aligned}$$

This proves (b). \square

PROOF OF LEMMA 3.4. It is a well-known fact that if $\omega \in \mathfrak{o}_K^*$ is a unit of absolute value one, then ω is a root of unity. Thus an element $w \in W$ satisfying $w + \bar{w} = 0$ must lie in W_{tor} , the subgroup of torsion elements of W . Hence if $w \in W(\chi)$ and $w + \bar{w} = 0$, then $w \in W_{\text{tor}}(\chi)$. We will show $W_{\text{tor}}(\chi) = 0$.

Let $N_1 = \text{lcm}(2, N)$. Then N_1 is the number of roots of unity in the $\mathbf{Q}(e(1/N))$ and $W_{\text{tor}} \cong \mu_{N_1} \times \mu_m$, where μ_n denotes the group of n th roots of unity. Hence $W_{\text{tor}}(\chi) \cong \mu_{N_1}(\chi) \times \mu_m(\chi)$.

Since m acts invertibly on $\mu_m(\chi)$, we have $\mu_m(\chi) = (0)$.

The group of squares in the Galois group $G = \text{Gal}(K/\mathbf{Q}(e(1/N))) \cong (\mathbf{Z}/m\mathbf{Z})^*$ acts at the same time trivially and via χ on $\mu_{N_1}(\chi)$. Therefore $\mu_{N_1}(\chi)$ is annihilated by $\chi(r) - 1$ for every quadratic residue $r \in (\mathbf{Z}/m\mathbf{Z})^*$. Since χ is not quadratic, we can choose r so that $\chi(r)$ is a nontrivial $((m-1)/2)$ th root of unity. Hence $(m-1)/2$ annihilates $\mu_{N_1}(\chi)$. But the conditions $m \equiv 3 \pmod{4}$ and $m \equiv -1 \pmod{N}$ show that N_1 and $(m-1)/2$ are relatively prime. Thus $\mu_{N_1}(\chi) = 0$. \square

4. Subgroups of the cuspidal group. Let $N > 0$ and Γ be one of the groups $\Gamma_1(N)$ or $\Gamma_0(N)$. For a divisor $D \in \mathscr{D}_\Gamma^0 \otimes \mathbf{C}$ let $R(D)$ be the \mathbf{Z} -submodule of \mathbf{C} generated by the coefficients of D .

DEFINITION 4.1. The subgroup of C_Γ associated to D is the image of the composition

$$\begin{aligned} \text{Hom}(R(D); \mathbf{Z}) &\rightarrow \mathscr{D}_\Gamma^0 \rightarrow C_\Gamma \\ \phi &\mapsto \phi_*(D). \end{aligned} \quad \square$$

It follows from the definition that for $\lambda \in \mathbf{C}^*$ the subgroup associated to D is equal to the subgroup associated to λD .

The Hecke algebra $\mathbf{T} = \mathbf{Z}[T_l, \langle a \rangle: l \nmid N \text{ prime}, a \in (\mathbf{Z}/N\mathbf{Z})^*]$ and the Galois group $G = \text{Gal}(\mathbf{Q}(e(1/N))/\mathbf{Q})$ act on \mathscr{D}_Γ^0 . If $\mathbf{T}[G]$ acts on D via a character $\psi: \mathbf{T}[G] \rightarrow \mathbf{C}$, then the subgroup of C_Γ associated to D is a cyclic $\mathbf{T}[G]$ -submodule of C_Γ . In this section we show how Theorems 1.2 and 1.3 can be used to study this module. In case $\Gamma = \Gamma_1(N)$ we will obtain a complete description. When $\Gamma = \Gamma_0(N)$ we describe it only modulo the Shimura group.

For the rest of this section we will write Γ_1 for $\Gamma_1(N)$ and Γ_0 for $\Gamma_0(N)$.

Over Γ_1 our method may be summarized in two steps:

(1) Find a weight two Eisenstein series E over Γ_1 with $\delta_{\Gamma_1}(E) = D$.

(2) Use Theorem 1.3 to find $\mathscr{P}_{\Gamma_1}(E)$.

Then the subgroup of C_{Γ_1} associated to D is $C_{\Gamma_1}(E)$ and its structure is described in Theorem 1.2.

The first step is easily accomplished using Proposition 4.7(d). To carry out (2) we need some divisibility properties of the values $\Lambda_{\pm}(E, \chi, 1)$. Proposition 4.7(c) allows us to give explicit formulas for $\Lambda_{\pm}(E, \chi, 1)$ in terms of generalized Bernoulli numbers and Euler factors. The following theorem then provides the necessary divisibility information.

THEOREM 4.2. *Let ε be a primitive Dirichlet character of conductor M and $m \nmid M$ an odd prime. Let \mathfrak{X}^+ (resp. \mathfrak{X}^-) be the set of primitive Dirichlet characters χ whose conductors are powers of m and satisfy $\chi(-1) = 1$ (resp. $\chi(-1) = -1$). Let $\mathfrak{P} \nmid m$ be a prime of $\overline{\mathbf{Q}}$ and \pm be a choice of sign. Then*

(a) *For every algebraic integer α and prime number l , there are infinitely many $\chi \in \mathfrak{X}^{\pm}$ such that*

$$\mathfrak{P} \nmid (1 - \alpha\chi(l));$$

(b) *If $m \equiv 3 \pmod{4}$ and $\chi \in \mathfrak{X}^{\pm}$ is not quadratic, then*

$$\tfrac{1}{2}\mathbf{B}_1(\varepsilon\chi) \in \mathbf{Z}[\varepsilon, \chi, \tfrac{1}{m}];$$

(c) (*L. Washington [20]*) *For all but finitely many $\chi \in \mathfrak{X}^{-\varepsilon(-1)}$, $\tfrac{1}{2}\mathbf{B}_1(\varepsilon\chi)$ is a \mathfrak{P} -unit.*

PROOF. (a) For any m -power root of unity, ζ , there is a $\chi \in \mathfrak{X}^+$ such that $\chi(l) = \zeta$. Since the m -power roots of unity are distinct modulo \mathfrak{P} , there is at most one choice of ζ for which $\mathfrak{P} \mid (1 - \alpha\zeta)$. This proves the result for \mathfrak{X}^+ . Fix $\chi_0 \in \mathfrak{X}^-$. Then there are infinitely many $\chi \in \mathfrak{X}^+$ such that $\mathfrak{P} \nmid (1 - \alpha\chi_0(l)\chi(l))$. Since $\chi_0\chi$ is imprimitive for only finitely many $\chi \in \mathfrak{X}^+$, the result follows.

(b) If $\varepsilon\chi(-1) = 1$, then $\mathbf{B}_1(\varepsilon\chi) = 0$. If $\varepsilon\chi(-1) = -1$, then $\varepsilon\chi(-1)(-1/m) = 1$. Thus

$$\mathbf{B}_1(\varepsilon\chi) = \mathbf{B}_1(\varepsilon\chi) + \mathbf{B}_1(\varepsilon\chi(\cdot/m))$$

$$= 2\varepsilon(m_{\chi})\chi(M) \cdot \sum_{\substack{a=0 \\ (a/m)=(M/m)}}^{m_{\chi}-1} \chi(a)\mathbf{B}_{1,\varepsilon}\left(\frac{Ma}{m_{\chi}}\right). \quad \square$$

In case $\Gamma = \Gamma_0$ we use the natural projection $\pi: X_1(N) \rightarrow X_0(N)$. The map π induces $\pi^*: \text{Pic}^0(X_0(N)) \rightarrow \text{Pic}^0(X_1(N))$ whose kernel Σ_N is known as the Shimura group. Let $X_2(N)$ be the maximal unramified extension of $X_0(N)$ intermediate to π . By Kummer theory there is a perfect pairing of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -modules

$$\text{Aut}(X_2(N)/X_0(N)) \times \Sigma_N \rightarrow \mu,$$

where μ is the group of roots of unity. Since the Nebentypus automorphisms of $X_1(N)$ are defined over \mathbf{Q} we see that $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts trivially on $\text{Aut}(X_2(N)/X_0(N))$. Thus Σ_N is a μ -type group.

Let $D \in \mathscr{D}_{\Gamma_0}^0 \otimes \mathbf{C}$ be a divisor on which $\mathbf{T}[G]$ acts via a character ψ . Using Proposition 4.7(d) we can find $E \in \mathscr{E}_{\Gamma_1}$ so that $\delta_{\Gamma_1}(E) = \pi^*D$. Then $E \in \mathscr{E}_{\Gamma_0}$ and $\delta_{\Gamma_0}(E) = D$. Using Theorems 1.3 and 4.2 we can calculate $\mathscr{P}_{\Gamma_1}(E)$ as before. Let

$$(4.1) \quad \begin{aligned} C_{\Gamma_0}^{(s)}(E) &= \pi^*(C_{\Gamma_0}(E)), \\ A_{\Gamma_0}^{(s)}(E) &= (\mathscr{P}_{\Gamma_1}(E) + \mathscr{R}_{\Gamma_0}(E))/\mathscr{R}_{\Gamma_0}(E). \end{aligned}$$

Clearly $C_{\Gamma_0}^{(s)}(E) \subseteq C_{\Gamma_1}(E)$. The following proposition will allow us to describe the structure of $C_{\Gamma_0}^{(s)}(E)$ as a $\mathbb{T}[G]$ -module.

PROPOSITION 4.3. *There is a canonical perfect pairing*

$$C_{\Gamma_0}^{(s)}(E) \times A_{\Gamma_0}^{(s)}(E) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

PROOF. $C_{\Gamma_0}^{(s)}(E)$ is the image of the composition

$$\begin{aligned} \mathrm{Hom}(\mathcal{R}_{\Gamma_0}(E); \mathbf{Z}) &\rightarrow \mathcal{D}_{\Gamma_0}^0 \xrightarrow{\pi^*} \mathcal{D}_{\Gamma_1}^0 \rightarrow C_{\Gamma_1} \\ \phi &\mapsto \phi_*(D). \end{aligned}$$

Since any $\phi \in \mathrm{Hom}(\mathcal{R}_{\Gamma_0}(E); \mathbf{Z})$ extends uniquely to a homomorphism

$$\phi: (\mathcal{P}_{\Gamma_1}(E) + \mathcal{R}_{\Gamma_0}(E)) \rightarrow \mathbf{Q}$$

we obtain a natural pairing

$$(4.2) \quad \mathrm{Hom}(\mathcal{R}_{\Gamma_0}(E); \mathbf{Z}) \times (\mathcal{P}_{\Gamma_1}(E) + \mathcal{R}_{\Gamma_0}(E)) \rightarrow \mathbf{Q}.$$

An element $\phi \in \mathrm{Hom}(\mathcal{R}_{\Gamma_0}(E); \mathbf{Z})$ goes to zero in C_{Γ_1} if and only if $\pi^*(\phi_*D) = \phi_*(\pi^*D) = \phi_*(\delta_{\Gamma_1}(E))$ is principal if and only if $\phi(\mathcal{P}_{\Gamma_1}(E)) \subseteq \mathbf{Z}$. Thus (4.2) induces a perfect pairing

$$C_{\Gamma_0}^{(s)}(D) \times A_{\Gamma_0}^{(s)}(E) \rightarrow \mathbf{Q}/\mathbf{Z}$$

as desired. \square

Since the Shimura group Σ_N can be computed explicitly and since we have an exact sequence

$$(4.3) \quad 0 \rightarrow \Sigma_N \cap C_{\Gamma_0}(E) \rightarrow C_{\Gamma_0}(E) \rightarrow C_{\Gamma_0}^{(s)}(E) \rightarrow 0,$$

we can often use our complete knowledge of $C_{\Gamma}^{(s)}(E)$ to say a good deal about $C_{\Gamma_0}(E)$.

Before proceeding to a general description of the cusps and the Eisenstein series, we illustrate these techniques with a familiar example.

EXAMPLE 4.4. Let $\Gamma_0 = \Gamma_0(N)$, where N is prime. There are only two cusps (0) and $(i\infty)$ on $X_0(N)$, thus C_{Γ_0} is generated by the class of $D = (0) - (i\infty)$. The space of Eisenstein series over Γ_0 is spanned by

$$E(z) = \frac{N-1}{24} + \sum_{n=1}^{\infty} \sigma_N(n) e(nz),$$

where $\sigma_N(n) = \sum_{d|n; N \nmid d} d$. Thus $C_{\Gamma_0} = C_{\Gamma_0}(E)$.

By Theorem 1.3(a), $r_{\Gamma_0, E}(i\infty) = (N-1)/24$. Since $\delta_{\Gamma_0}(E)$ has degree zero we have $\delta_{\Gamma_0}(E) = (1-N)/24 \cdot D$. Thus $\mathcal{R}_{\Gamma_0}(E) = \frac{N-1}{24} \mathbf{Z}$.

The L -function of E is $L(E, s) = (1 - N^{1-s})\zeta(s)\zeta(s-1)$. Let $S = \{m \text{ prime} \mid m \equiv -1 \pmod{4N}\}$. Then S satisfies (1.8). If $m \in S$ and χ is an odd primitive character with conductor a power of m , then

$$\Lambda(E, \chi, 1) = \frac{1}{2}(1 - \chi(N))\mathbf{B}_1(\chi)\mathbf{B}_1(\bar{\chi})$$

and

$$\Lambda_{\pm}(E, \chi, 1) = (1 - \chi(N)) \cdot \frac{\mathbf{B}_1(\chi)}{2} \cdot \frac{\mathbf{B}_1(\bar{\chi})}{2}.$$

Thus Theorems 4.2(b) and 1.3 show $\mathscr{P}_{\Gamma_1(N)}(E) \subseteq \mathbf{Z} + \frac{N-1}{24}\mathbf{Z}$. By Theorem 4.2(a), (c) we have equality, $\mathscr{P}_{\Gamma_1(N)}(E) = \mathbf{Z} + \frac{N-1}{24}\mathbf{Z}$. Thus

$$A_{\Gamma_0}^{(s)}(E) = (\mathbf{Z} + \frac{N-1}{24}\mathbf{Z})/\mathbf{Z} \cong \mathbf{Z}/n'\mathbf{Z},$$

where $n' = \text{Num}((N - 1)/24)$. By Proposition 4.3 $C_{\Gamma_0}^{(s)}(E) \cong \mathbf{Z}/n'\mathbf{Z}$.

We therefore have an exact sequence

$$0 \rightarrow \Sigma_N \cap C_{\Gamma_0} \rightarrow C_{\Gamma_0} \rightarrow \mathbf{Z}/n'\mathbf{Z} \rightarrow 0.$$

The group Σ_N is cyclic of order $n = \text{Num}((N - 1)/12)$. Thus $\Sigma_N \cong \mu_n$. Since G acts trivially on C_{Γ_0} (Theorem 1.2(b)) we have

$$\Sigma_N \cap C_{\Gamma_0} \cong (0) \quad \text{or} \quad (\mathbf{Z}/2\mathbf{Z}).$$

To decide which of these possibilities occurs we can appeal to the explicit formulas for the cocycle of E [19] and then use classical congruence formulas modulo 8 for Dedekind sums [16, p. 34] to show

$$\mathscr{P}_{\Gamma_0(N)}(E) = \begin{cases} \frac{1}{2}\mathbf{Z} + \frac{N-1}{24}\mathbf{Z} & \text{if } N \equiv 1 \pmod{8}, \\ \mathbf{Z} + \frac{N-1}{24}\mathbf{Z} & \text{otherwise.} \end{cases}$$

Thus

$$C_{\Gamma_0} \cong \mathbf{Z}/n\mathbf{Z}, \quad n = \text{Num}((N - 1)/12). \quad \square$$

Fix $N > 0$. We will use Shimura’s notation for the cusps on $X(N)$. For $a, b \in \mathbf{Z}$ such that the triple (a, b, N) is relatively prime, $\begin{bmatrix} a \\ b \end{bmatrix}_{\Gamma(N)}$ denotes the cusp represented by $\alpha/\beta \in \mathbf{Q}$ with $\alpha, \beta \neq 0$ relatively prime integers satisfying $\alpha \equiv a, \beta \equiv b \pmod{N}$. The group $\text{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts on $\text{cusps}(\Gamma(N))$ by matrix multiplication.

Let Γ be one of the groups $\Gamma_1 = \Gamma_1(N)$, or $\Gamma_0 = \Gamma_0(N)$. The natural map $\Gamma \rightarrow \text{GL}_2(\mathbf{Z}/N\mathbf{Z})$ affords an action of Γ on $\text{cusps}(\Gamma(N))$. The cusps on X_Γ may be identified with the Γ -orbits of cusps on $X(N)$. Let $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma$ be the cusp represented by the Γ -orbit of $\begin{bmatrix} a \\ b \end{bmatrix}_{\Gamma(N)}$. Then we have the relations

(4.4)

(i)

(ii)

(iii)

$\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} a' \\ b' \end{bmatrix}_\Gamma \quad \text{if } a \equiv a', b \equiv b' \pmod{N};$
 $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} a' \\ b \end{bmatrix}_\Gamma \quad \text{if } a \equiv a' \pmod{b};$
 $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} -a \\ -b \end{bmatrix}_\Gamma.$

If $\Gamma = \Gamma_0$, (iii) may be strengthened to

$$(iii)' \quad \begin{bmatrix} a \\ b \end{bmatrix}_{\Gamma_0} = \begin{bmatrix} a \\ rb \end{bmatrix}_{\Gamma_0} \quad \text{for all } r \in \mathbf{Z} \text{ with } (r, N) = 1.$$

The integer $d = (b, N)$ depends only on the cusp $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma$. We will refer to d as the “divisor” of $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma$. The ramification index of $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma$ over $X(1)$ is given by

(4.5)

$e_\Gamma\left(\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma\right) = \begin{cases} \frac{N}{d} & \text{if } \Gamma = \Gamma_1(N), \\ \frac{N}{dt} & \text{if } \Gamma = \Gamma_0(N), \end{cases}$

where $t = \text{gcd}(d, N/d)$.

Let $T(N) = \{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}_N \} \subseteq \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ be the standard torus. Then $T(N)$ normalizes the image of Γ in $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ and therefore acts on $\mathrm{cusps}(\Gamma)$. The action is given by

$$(4.6) \quad \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}_N \cdot \begin{bmatrix} a \\ b \end{bmatrix}_\Gamma = \begin{bmatrix} ra \\ sb \end{bmatrix}_\Gamma$$

for $r, s \in (\mathbf{Z}/N\mathbf{Z})^*$.

For a divisor d of N let $\mathcal{D}_{\Gamma, d} \subseteq \mathcal{D}_\Gamma$ be the subgroup of divisors supported on cusps with divisor d . The above formulas show that $\mathcal{D}_{\Gamma, d}$ is preserved by $T(N)$.

In Shimura's canonical model for X_Γ over \mathbf{Q} , the cusps are defined over $\mathbf{Q}(e(1/N))$. If we fix the isomorphism

$$(4.7) \quad (\mathbf{Z}/N\mathbf{Z})^* \rightarrow G = \mathrm{Gal}(\mathbf{Q}(e(1/N))/\mathbf{Q}) \\ r \mapsto (\tau_r: e(1/N) \mapsto e(r/N)),$$

then the action of G on \mathcal{D}_Γ is given in terms of the action of $T(N)$ by the correspondence

$$(4.8) \quad \tau_r \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix}_N$$

[19, Theorem 1.3.1]. Hence the rationality properties of the cusps are summarized by

$$(4.9) \quad \begin{bmatrix} a \\ b \end{bmatrix}_\Gamma \text{ is rational over } \begin{cases} \mathbf{Q}(e(d/N)) & \text{if } \Gamma = \Gamma(N), \\ \mathbf{Q}(e(1/t)) & \text{if } \Gamma = \Gamma_0(N), \end{cases}$$

where d is the divisor of $\begin{bmatrix} a \\ b \end{bmatrix}_\Gamma$ and $t = \gcd(d, N/d)$.

The action on \mathcal{D}_Γ of the Nebentypus automorphisms $\langle r \rangle$, $r \in (\mathbf{Z}/N\mathbf{Z})^*$, and the Hecke operators T_l , l prime, $l \nmid N$, can also be described in terms of the action of $T(N)$ by

$$(4.10) \quad \langle r \rangle \leftrightarrow \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}_N, \\ T_l \leftrightarrow \tau_l + l \langle l \rangle \tau_l^{-1}.$$

We warn the reader that this action of $\langle r \rangle$, T_l on \mathcal{D}_Γ is dual to what may be considered the standard action. This is due to the fact that we are viewing C_Γ as a subgroup of the *contravariant* object $\mathrm{Pic}^0(X_\Gamma)$. For $E \in \mathcal{E}_\Gamma$ define $E|\langle r \rangle$, $E|T_l$ as usual by

$$(4.11) \quad E|\langle r \rangle = E|\gamma \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ with } d \equiv r \pmod{N}; \\ E|T_l = \sum_{k=0}^{l-1} E \left| \begin{pmatrix} 1 & k \\ 0 & l \end{pmatrix} \right| + E|\langle l \rangle \left| \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \right|.$$

Then with our conventions for the action of $\langle r \rangle$, T_l on \mathcal{D}_Γ we have

$$(4.12) \quad \delta_\Gamma(E|\langle r \rangle) = \langle r \rangle \cdot \delta_\Gamma(E), \quad \delta_\Gamma(E|T_l) = T_l \cdot \delta_\Gamma(E).$$

Fix a character $\psi: T(N) \rightarrow \mathbf{C}^*$. Then there are primitive Dirichlet characters $\varepsilon_1, \varepsilon_2$ of conductors N_1, N_2 divisors of N such that

$$(4.13) \quad \psi \left(\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}_N \right) = \bar{\varepsilon}_1(s) \varepsilon_2(r)$$

for $r, s \in (\mathbf{Z}/N\mathbf{Z})^*$. Let $\varepsilon = \varepsilon_1 \varepsilon_2$. Then ε need not be primitive.

For a divisor d of N let $\mathcal{D}_{\Gamma,d}(\psi)$ be the $\mathbf{Z}[\psi]$ -submodule of $\mathcal{D}_{\Gamma,d} \otimes \mathbf{Z}[\psi]$ on which $T(N)$ acts via ψ and let $\mathcal{D}_{\Gamma}(\psi) = \sum_{d|N} \mathcal{D}_{\Gamma,d}(\psi)$. In particular, for $r \in (\mathbf{Z}/N\mathbf{Z})^*$, $\tau_r \in G$ acts on $\mathcal{D}_{\Gamma}(\psi)$ as $\varepsilon_1(r)$, and $\langle r \rangle$ acts as $\varepsilon(r)$. For a prime $l \nmid N$, T_l acts as $\varepsilon_1(l) + l\varepsilon_2(l)$. By (4.4)(iii) $\mathcal{D}_{\Gamma}(\psi) = 0$ unless $\varepsilon(-1) = 1$.

PROPOSITION 4.5. *Let ψ be as above, with $\varepsilon(-1) = 1$.*

(a) $\mathcal{D}_{\Gamma,d}(\psi) \neq 0$ if and only if $N_2|d$, $N_1|(N/d)$ and, in case $\Gamma = \Gamma_0(N)$, $\varepsilon_1 = \bar{\varepsilon}_2$.

(b) If $\mathcal{D}_{\Gamma,d}(\psi) \neq 0$, then it is generated as a $\mathbf{Z}[\psi]$ -module by

$$D_{\Gamma,d}(\psi) = \sum \varepsilon_1(b) \bar{\varepsilon}_2(a) \left[\frac{a}{db} \right]_{\Gamma},$$

where the sum is over cusps $\left[\frac{a}{db} \right]_{\Gamma}$ with divisor d .

The proof is a straightforward calculation which uses the primitivity of $\varepsilon_1, \varepsilon_2$. \square

For a character $\psi: T(N) \rightarrow \mathbf{C}^*$ let $\mathcal{D}_{\Gamma}^0(\psi)$ be the degree zero divisors in $\mathcal{D}_{\Gamma}(\psi)$. Let $C_{\Gamma}(\psi) \subseteq C_{\Gamma}$ be the image of the composition

$$\begin{aligned} \mathcal{D}_{\Gamma}^0(\psi) \otimes_{\mathbf{Z}[\psi]} \text{Hom}(\mathbf{Z}[\psi]; \mathbf{Z}) &\rightarrow \mathcal{D}_{\Gamma}^0 \rightarrow C_{\Gamma} \\ D \otimes \phi &\mapsto \phi_*(D). \end{aligned}$$

In case ψ is nontrivial we have $\mathcal{D}_{\Gamma}^0(\psi) = \mathcal{D}_{\Gamma}(\psi)$. In this case we let $C_{\Gamma,d}(\psi) \subseteq C_{\Gamma}(\psi)$ be the image of $\mathcal{D}_{\Gamma,d}(\psi) \otimes \text{Hom}(\mathbf{Z}[\psi]; \mathbf{Z})$ under the above homomorphism. Then we have

$$C_{\Gamma}(\psi) = \sum_{d|N} C_{\Gamma,d}(\psi),$$

though this sum is usually not direct.

It is convenient to extend the space \mathcal{E} of weight two Eisenstein series of all levels by adjoining the nonholomorphic Eisenstein series of level one:

$$\phi_{(0,0)}(z) = \frac{1}{2\pi i(z - \bar{z})} + \frac{1}{12} - 2 \cdot \sum_{k=1}^{\infty} k \cdot \sum_{m=1}^{\infty} e(mkz).$$

Let $\mathcal{E}^* = \mathcal{E} \oplus \mathbf{C} \cdot \phi_0$. A typical element $E \in \mathcal{E}^*$ has a Fourier expansion of the form

$$(4.14) \quad E(z) = \frac{a_{-1}(E)}{2\pi i(z - \bar{z})} + a_0(E) + \sum_{k=1}^{\infty} a_k(E) e\left(\frac{kz}{N}\right).$$

The space \mathcal{E}^* is a module under the weight two action of the group $\text{GL}_2^+(\mathbf{Q})$ of 2×2 rational matrices with positive determinant. For a congruence group $\Gamma \subseteq \text{SL}_2(\mathbf{Z})$ let $\mathcal{E}_{\Gamma}^* = \mathcal{E}_{\Gamma} \oplus \mathbf{C} \cdot \phi_0$ be the space of Γ -invariants in \mathcal{E}^* . If $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$, then the action of $T(N)$ on \mathcal{E}_{Γ} extends trivially to an action on \mathcal{E}_{Γ}^* .

We may extend δ_{Γ} to a linear map $\delta_{\Gamma}: \mathcal{E}_{\Gamma}^* \rightarrow \mathcal{D}_{\Gamma} \otimes \mathbf{C}$ by defining

$$\delta_{\Gamma}(\phi_0) = \frac{1}{12} \sum_{x \in \text{cusps}(\Gamma)} e_{\Gamma}(x) \cdot (x).$$

We also extend the definition of $L(E, \chi, s)$ to all $E \in \mathcal{E}^*$ by setting

$$L(\phi_0, \chi, s) = -2L(s, \chi)L(s-1, \chi)$$

and extending linearly. We then define $\Lambda(E, \chi, 1)$, $\Lambda_{\pm}(E, \chi, 1)$ as before (1.6), (1.7). The basic properties (1.4), (4.11) of these symbols are preserved by the extension.

As before let $\Gamma_1 = \Gamma_1(N)$, $\Gamma_0 = \Gamma_0(N)$ and $\psi: T(N) \rightarrow \mathbf{C}^*$ be a character associated to $\varepsilon_1, \varepsilon_2$ of conductors N_1, N_2 as in (4.13). Assume $\varepsilon = \varepsilon_1 \varepsilon_2$ is an even character, and $(N_1 N_2) | N$.

DEFINITION 4.6. (a)

$$E_{(\varepsilon_1, \varepsilon_2)} = \frac{1}{2} \sum_{r=0}^{N_1-1} \sum_{s=0}^{N_2-1} \bar{\varepsilon}_1(r) \varepsilon_2(s) \phi_{(s/N_2, r/N_1)};$$

(b) For d a divisor of N/N_1 such that $N_2 | d$ let

$$E_d(\psi) = E_{(\varepsilon_1, \varepsilon_2)} \left| \prod_{p|(N/d)} \left(1 - \frac{\varepsilon_1(p)}{p} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \prod_{p|d} \left(1 - \frac{\bar{\varepsilon}_2(p)}{p} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right) \cdot \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \right|,$$

where the product on the right is multiplication in the group ring $\mathbf{C}[\mathrm{GL}_2^+(\mathbf{Q})]$. \square

PROPOSITION 4.7. Let d be a divisor of N/N_1 such that $N_2 | d$ and let $E = E_d(\psi)$. Then

(a) $E \in \mathcal{E}_{\Gamma_1}^*$; $E| \langle r \rangle = \varepsilon(r)E$ for $r \in (\mathbf{Z}/N\mathbf{Z})^*$; $E|T_l = (\varepsilon_1(l) + l\varepsilon_2(l))E$ for prime $l \nmid N$.

(b)

$$\begin{aligned} L(E, s) &= -\tau(\bar{\varepsilon}_1) \left(\frac{d}{N_2} \right)^{1-s} \prod_{p|(N/d)} (1 - \varepsilon_1(p) p^{-s}) \\ &\quad \times \prod_{p|d} (1 - \bar{\varepsilon}_2(p) p^{s-2}) L(\varepsilon_1, s) L(\varepsilon_2, s-1). \end{aligned}$$

(c) For a primitive Dirichlet character χ of conductor m_χ with $(m_\chi, N) = 1$,

$$\begin{aligned} \Lambda(E, \chi, 1) &= -\frac{1}{2} \varepsilon_1(m_\chi) \chi \left(\frac{dN_1}{N_2} \right) \prod_{p|(N/d)} \left(1 - \frac{\varepsilon_1 \chi(p)}{p} \right) \\ &\quad \times \prod_{p|d} \left(1 - \frac{\bar{\varepsilon}_2 \bar{\chi}(p)}{p} \right) \mathbf{B}_1(\bar{\varepsilon}_1 \bar{\chi}) \mathbf{B}_1(\varepsilon_2 \chi). \end{aligned}$$

(d)

$$\delta_{\Gamma_1}(E) = \frac{\varepsilon_1(-1)N}{4n} \prod_{p|N} \left(1 - \frac{\xi(p)}{p^2} \right) \frac{\tau(\bar{\varepsilon}_1) \tau(\varepsilon_2)}{\tau(\bar{\xi})} \mathbf{B}_2(\bar{\xi}) \cdot D_{\Gamma_1, d}(\psi),$$

where ξ is the primitive Dirichlet character associated to $\varepsilon_1 \bar{\varepsilon}_2$, and n is the conductor of ξ .

The proof is given in the next section.

EXAMPLE 4.8. If $\Gamma_1 = \Gamma_1(N)$, $d|(N/N_1)$ and $N_2 | d$, then

$$C_{\Gamma_1, d}(\psi) \cong \mathbf{Z}[\psi] / \mathrm{Num}(\beta)$$

as $\mathbf{Z}[\psi]$ -modules. The action of $\mathbf{T}[G]$ is given in Theorem 1.2. Here $\mathrm{Num}(\beta)$ is the ideal $(\beta) \cap \mathbf{Z}[\psi]$, and

$$\beta = \frac{M \cdot N}{4n} \prod_{p|N} \left(1 - \frac{\xi(p)}{p^2} \right) \frac{\tau(\bar{\varepsilon}_1) \tau(\varepsilon_2)}{\tau(\bar{\xi})} \mathbf{B}_2(\bar{\xi}),$$

where $M = (\prod_{p|(N/d); p \nmid N_1} p)(\prod_{p|d; p \nmid N_2} p)$, ξ is the primitive Dirichlet character associated to $\varepsilon_1 \bar{\varepsilon}_2$, and n is the conductor of ξ .

PROOF. We have $C_{\Gamma_1, d}(\psi) = C_{\Gamma_1}(E)$, where $E = E_d(\psi)$. Proposition 4.7(d) shows $\mathcal{R}_{\Gamma_1}(E) = \frac{\beta}{M} \mathbf{Z}[\psi]$.

Let S be a set of primes satisfying (1.8). If $m \in S$ and χ has conductor a power of m , then

$$\begin{aligned} \Lambda_{\pm}(E, \chi, 1) &= -\varepsilon_1(m_{\chi}) \chi \left(\frac{dN_1}{N_2} \right) \prod_{p|(N/d)} \left(1 - \frac{\varepsilon_1 \chi(p)}{p} \right) \\ &\quad \times \prod_{p|d} \left(1 - \frac{\bar{\varepsilon}_1 \bar{\chi}(p)}{p} \right) \frac{\mathbf{B}_1(\bar{\varepsilon}_1 \bar{\chi})}{2} \frac{\mathbf{B}_1(\varepsilon_2 \chi)}{2}. \end{aligned}$$

By Theorem 4.2(b) $M \cdot \Lambda_{\pm}(E, \chi, 1) \in \mathbf{Z}[\psi, \chi, 1/m_{\chi}]$ for every $\chi \in \mathfrak{X}_S^{-\varepsilon_1(-1)}$. For $m \in S$ and $\mathfrak{p} \nmid m$ a prime of \mathbf{Q} , Theorem 4.2(a), (c) assures the existence of χ with conductor a power of m such that $M \cdot \Lambda_{\pm}(E, \chi, 1)$ is a \mathfrak{p} -unit. Thus by Theorem 1.3(b)

$$\mathcal{P}_{\Gamma_1}(E) = \frac{1}{M} \mathbf{Z}[\psi] + \mathcal{R}_{\Gamma_1}(E) = \frac{1}{M} \mathbf{Z}[\psi] + \frac{\beta}{M} \mathbf{Z}[\psi].$$

Then

$$\begin{aligned} A_{\Gamma_1}(E) &= \mathcal{P}_{\Gamma_1}(E) / \mathcal{R}_{\Gamma_1}(E) \\ &= \left(\frac{1}{M} \mathbf{Z}[\psi] + \frac{\beta}{M} \mathbf{Z}[\psi] \right) / \left(\frac{\beta}{M} \mathbf{Z}[\psi] \right) \cong \mathbf{Z}[\psi] / \text{Num}(\beta) \end{aligned}$$

and the result follows from Theorem 1.2. \square

EXAMPLE 4.9. Let $N = m^2$ and α be a primitive Dirichlet character of conductor m . Let ψ be the character of $T(N)$ defined by $\psi((\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix})_N) = \bar{\alpha}(rs)$. Let $\mathfrak{p} \subseteq \mathbf{Z}[\psi]$ be a prime ideal such that α is not the cyclotomic character at \mathfrak{p} (i.e. there exists $n \in \mathbf{Z}$ such that $\alpha(n) \not\equiv n \pmod{\mathfrak{p}}$). Then

$$C_{\Gamma_0(N)}(\psi)_{\mathfrak{p}} \cong (\mathbf{Z}[\alpha] / \text{Num}(\beta))_{\mathfrak{p}},$$

where

$$\beta = \frac{m}{4n} \prod_{l|m} \left(1 - \frac{\xi(l)}{l^2} \right) \frac{\tau(\bar{\alpha})^2}{\tau(\bar{\xi})} \mathbf{B}_2(\bar{\xi}),$$

ξ is the primitive character associated to α^2 , n is the conductor of ξ , and the subscript \mathfrak{p} denotes completion at \mathfrak{p} .

PROOF. Let $E = E_m(\psi) \in \mathcal{E}_{\Gamma_0}$. Then $C_{\Gamma_0}(\psi) = C_{\Gamma_0}(E)$. In the notation of (4.13) $\varepsilon_1 = \alpha_1$, $\varepsilon_2 = \bar{\alpha}$.

As in Example 4.8, $\mathcal{R}_{\Gamma_1}(E) = m\beta \mathbf{Z}[\alpha]$ and $\mathcal{P}_{\Gamma_1}(E) = \mathbf{Z}[\alpha] + \mathcal{R}_{\Gamma_1}(E) = \mathbf{Z}[\alpha] + m\beta \mathbf{Z}[\alpha]$. By (4.5) $\mathcal{R}_{\Gamma_0}(E) = \frac{1}{m} \mathcal{R}_{\Gamma_1}(E) = \beta \cdot \mathbf{Z}[\alpha]$. Thus

$$\begin{aligned} A_{\Gamma_0}^{(s)}(E) &= (\mathcal{P}_{\Gamma_1}(E) + \mathcal{R}_{\Gamma_0}(E)) / \mathcal{R}_{\Gamma_0}(E) \\ &= (\mathbf{Z}[\alpha] + \beta \mathbf{Z}[\alpha]) / \beta \mathbf{Z}[\alpha] = \mathbf{Z}[\alpha] / \text{Num}(\beta). \end{aligned}$$

Since α is not cyclotomic at \mathfrak{p} , we have $(\Sigma_N \cap C_{\Gamma_0}(E))_{\mathfrak{p}} = (0)$. Hence $C_{\Gamma_0}(\psi)_{\mathfrak{p}} = C_{\Gamma_0}(E)_{\mathfrak{p}} = C_{\Gamma_0}^{(s)}(E)_{\mathfrak{p}}$ and the result follows. \square

EXAMPLE 4.10. Let C be the p -primary component of the cuspidal group $C_{\Gamma_0(p^2)}$ on $X_0(p^2)$, where p is prime. The cusps are rational over $\mathbf{Q}(e(1/p))$ by (4.9). We identify $(\mathbf{Z}/p\mathbf{Z})^*$ and $G = \text{Gal}(\mathbf{Q}(e(1/p))/\mathbf{Q})$ as in (4.7). For each integer k with $0 < k < p - 1$ let $C^{(k)} \subseteq C$ be the subgroup on which G acts via ω^k , where $\omega: G \rightarrow \mathbf{Z}_p^*$ is the cyclotomic character defined by $\omega(r) \equiv r \pmod{p}$. Then

$$C^{(k)} \cong \begin{cases} (0) & \text{if } k = 1, (p-1)/2, (p+1)/2, \\ \mathbf{Z}_p/\mathbf{B}_2(\omega^{-2k}) & \text{if } 1 < k < (p-1)/2, \\ \mathbf{Z}_p/p\mathbf{B}_2(\omega^{-2k}) & \text{if } (p+1)/2 < k < p-1. \end{cases}$$

PROOF. We may assume $p > 3$. Fix k with $0 < k < p - 1$, a prime ideal $\mathfrak{p} \subseteq \mathbf{Z}[e(1/(p-1))]$ dividing p , and a primitive Dirichlet character α of conductor p such that $\alpha \equiv \omega^k \pmod{\mathfrak{p}}$. Let ψ, E, β be as in Example 4.9. One easily checks

$$C_{\Gamma_0}(\psi)_{\mathfrak{p}} \cong C^{(k)}.$$

As in Example 4.9 we have $C_{\Gamma_0}(\psi) = C_{\Gamma_0}(E)$. The order of Σ_{p^2} is easily seen to be $\text{Num}((p-1)/12)$. Hence $C_{\Gamma_0}(E)_{\mathfrak{p}} \cong C_{\Gamma_0}^{(s)}(E)_{\mathfrak{p}}$ even when α is cyclotomic at \mathfrak{p} . Thus

$$C^{(k)} \cong (\mathbf{Z}[\alpha]/\text{Num}(\beta))_{\mathfrak{p}}$$

by Example 4.9.

To calculate $\text{ord}_{\mathfrak{p}}(\beta)$ we use the following facts:

$$\begin{aligned} \text{ord}_{\mathfrak{p}}(p/n) &= \begin{cases} 0 & \text{if } k \neq (p-1)/2, \\ 1 & \text{if } k = (p-1)/2, \end{cases} \\ \text{ord}_{\mathfrak{p}}((1 - \xi(p)/p^2)) &= \begin{cases} 0 & \text{if } k \neq (p-1)/2, \\ -2 & \text{if } k = (p-1)/2, \end{cases} \\ \text{ord}_{\mathfrak{p}}(\tau(\bar{\alpha})^2/\tau(\bar{\xi})) &= \begin{cases} 0 & \text{if } 0 < k < (p-1)/2, \\ 1 & \text{if } (p-1)/2 \leq k < p-1 \quad [9], \end{cases} \\ \text{ord}_{\mathfrak{p}}(\mathbf{B}_2(\bar{\xi})) &= \begin{cases} -1 & \text{if } k = 1, (p+1)/2, \\ 0 & \text{if } k = (p-1)/2, \\ \geq 0 & \text{otherwise} \quad [9]. \end{cases} \end{aligned}$$

We conclude

$$\text{ord}_{\mathfrak{p}}(\beta) = \begin{cases} -1 & \text{if } k = 1, \\ 0 & \text{if } k = (p-1)/2, (p+1)/2, \\ \text{ord}_{\mathfrak{p}}(\mathbf{B}_2(\bar{\xi})) \geq 0 & \text{if } 1 < k < (p-1)/2, \\ 1 + \text{ord}_{\mathfrak{p}}(\mathbf{B}_2(\bar{\xi})) \geq 1 & \text{if } (p+1)/2 < k < p-1. \end{cases}$$

Since $\mathbf{Z}[\alpha]_{\mathfrak{p}} \cong \mathbf{Z}_p$ and $\mathbf{B}_2(\bar{\xi})$ is sent to $\mathbf{B}_2(\omega^{-2k})$ under this isomorphism, the result follows. \square

5. Proof of Proposition 4.7. (a) Using (3.9) one easily sees the equality $E_{(\epsilon_1, \epsilon_2)}| \gamma = E_{(\epsilon_1, \epsilon_2)}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b \equiv 0 \pmod{N_2}$, $c \equiv 0 \pmod{N_1}$ and $a \equiv d \equiv \pm 1 \pmod{N_1 N_2}$. Thus, if k is a divisor of N/N_1 such that $N_2|k$, then

$$E_{(\epsilon_1, \epsilon_2)} \Big| \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{E}_{\Gamma_1(N)}^*.$$

Since E is a linear combination of terms of this form, we have also $E \in \mathcal{E}_{\Gamma_1(N)}^*$. The other two statements follow from corresponding statements for $E_{(\epsilon_1, \epsilon_2)}$, or from an examination of the L -series in (b).

(b) A straightforward calculation [19, 3.4.2] using the Fourier expansion of the ϕ_x shows

$$L(E_{(\epsilon_1, \epsilon_2)}, s) = -\tau(\bar{\epsilon}_1) N_s^{s-1} L(\epsilon_1, s) L(\epsilon_2, s-1).$$

The L -series of E is then easily calculated using the identity

$$(5.1) \quad L\left(f \Big| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, s\right) = d^{1-s} L(f, s)$$

for $f \in \mathcal{E}^*$.

(c)

$$\begin{aligned} \Lambda(E, \chi, 1) &= \frac{\tau(\bar{\chi}) L(E, \chi, 1)}{2\pi i} \\ &= -\left(\frac{\tau(\bar{\chi})}{2\pi i}\right) \tau(\bar{\epsilon}_1) \chi\left(\frac{d}{N_2}\right) \prod_{p|(N/d)} \left(1 - \frac{\epsilon_1 \chi(p)}{p}\right) \\ &\quad \times \prod_{p|d} \left(1 - \frac{\bar{\epsilon}_2 \bar{\chi}(p)}{p}\right) L(\epsilon_1 \chi, 1) L(\epsilon_2 \chi, 0). \end{aligned}$$

The result follows from substitution with the identities:

$$\begin{aligned} L(\epsilon_1 \chi, 1) &= \frac{-\pi i}{\tau(\bar{\epsilon}_1 \bar{\chi})} \mathbf{B}_1(\bar{\epsilon}_1 \bar{\chi}), \\ L(\epsilon_2 \chi, 0) &= -\mathbf{B}_1(\epsilon_1 \chi), \\ \frac{\tau(\bar{\chi}) \tau(\bar{\epsilon}_1)}{\tau(\bar{\chi} \bar{\epsilon}_1)} &= \chi(N_1) \epsilon_1(m_\chi). \end{aligned}$$

(d) For $t \in T$ we have $E|t = \psi(t)^{-1}E$. Thus $t \cdot \delta_{\Gamma_1}(E) = \delta_{\Gamma_1}(E|t^{-1}) = \psi(t) \delta_{\Gamma_1}(E)$. It follows that $\delta_{\Gamma_1}(E)$ may be written as

$$(5.2) \quad \delta_{\Gamma_1}(E) = \sum c(n) \cdot D_{\Gamma_1, n}(\psi),$$

where the sum is over $n > 0$ satisfying

$$(5.3) \quad N_2|n, \quad n \Big| \frac{N}{N_1}.$$

We will calculate the coefficients $c(n)$ using an induction on N .

To start the induction suppose $N = N_1 N_2$. Then $E = E_{(\epsilon_1, \epsilon_2)} \left(\begin{smallmatrix} N_2 & 0 \\ 0 & 1 \end{smallmatrix} \right)$. There is only one n satisfying (5.3); hence

$$\delta_{\Gamma_1}(E) = c(N_2) \cdot D_{\Gamma_1, N_2}(\psi),$$

where (Theorem 1.3)

$$\begin{aligned} c(N_2) &= r_{\Gamma_1, E} \left(\begin{bmatrix} 1 \\ N_2 \end{bmatrix}_{\Gamma_1} \right) = e_{\Gamma_1} \left(\begin{bmatrix} 1 \\ N_2 \end{bmatrix}_{\Gamma_1} \right) \cdot a_0 \left(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right. \right) \\ &= N_1 \cdot a_0 \left(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right. \right). \end{aligned}$$

Thus

$$(5.4) \quad \delta_{\Gamma_1}(E) = N_1 \cdot a_0 \left(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right. \right) \cdot D_{\Gamma_1, N_2}(\psi).$$

We will calculate $a_0(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right.)$ using the identity $a_0(f) = -L(f, 0)$ for $f \in \mathcal{E}^*$. By expressing E in terms of the functions ϕ_x and applying $\begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix}$ it is easy to derive expressions for $a_0(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right.)$ and $L(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right., s)$. Unfortunately, these expressions require a great deal of work to simplify to the desired expressions. We therefore prefer the following trick.

If we compare the coefficient of $[a N_2]_{\Gamma_1}$, $a \in \mathbf{Z}$, on both sides of the equality (5.4) we find

$$a_0 \left(E \left| \begin{pmatrix} 1 & 0 \\ a N_2 & 1 \end{pmatrix} \right. \right) = \epsilon_1(a) a_0 \left(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right. \right).$$

Hence if we let $\nu(N_1) = \#((\mathbf{Z}/N_1\mathbf{Z})^*)$, then

$$(5.5) \quad \nu(N_1) a_0 \left(E \left| \begin{pmatrix} 1 & 0 \\ N_2 & 1 \end{pmatrix} \right. \right) = a_0(E'),$$

where

$$E' = \sum_{a=0}^{N_1-1} \bar{\epsilon}_1(a) E \left| \begin{pmatrix} 1 & 0 \\ a N_2 & 1 \end{pmatrix} \right|.$$

It therefore suffices to find $a_0(E')$.

For a primitive Dirichlet character χ of conductor n , let

$$R(\chi) = \sum_{a=0}^{n-1} \bar{\chi}(a) \cdot \begin{pmatrix} 1 & a/n \\ 0 & 1 \end{pmatrix} \in \mathbf{C}[\mathrm{GL}_2^+(\mathbf{Q})].$$

In the group ring $\mathbf{C}[\mathrm{GL}_2^+(\mathbf{Q})]$ we have the identity

$$(5.6) \quad \epsilon_1(-1) \sum_{a=0}^{N_1-1} \bar{\epsilon}_1(a) \begin{pmatrix} 1 & 0 \\ a N_2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ N_1 N_2 & 0 \end{pmatrix} R(\epsilon_1) \begin{pmatrix} 0 & -1 \\ N_1 N_2 & 0 \end{pmatrix}^{-1}.$$

We will calculate $L(E', s)$ using the following lemma. For $f \in \mathcal{E}^*$ let

$$(5.7) \quad D(f, s) = i \cdot \Gamma(s)(2\pi)^{-s} L(f, s).$$

LEMMA 5.1. For $f \in \mathcal{E}^*$ we have

$$(a) \ D(f| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s) + D(f, 2-s) = 0;$$

$$(b) \text{ For a primitive Dirichlet character } \chi \text{ and } f \in \mathcal{E}_{\Gamma_1(A)}^*, A > 0,$$

$$L(f|R(\chi), s) = \tau(\bar{\chi}) L(f, \chi, s);$$

$$(c) \text{ For primitive Dirichlet characters } \xi_1, \xi_2 \text{ of conductors } m_1, m_2,$$

$$E_{(\xi_1, \xi_2)} \Big| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = E_{(\bar{\xi}_2, \bar{\xi}_1)}.$$

PROOF. The functional equation in (a) is well known (e.g. [15]). The second statement (b) is a standard calculation. A simple calculation from Definition 4.6 using (3.9) proves (c). \square

Returning to the calculation of $L(E', s)$ let

$$E_1 = E \Big| \begin{pmatrix} 0 & -1 \\ N_1 N_2 & 0 \end{pmatrix}, \quad E_2 = E_1 | R(\epsilon_1), \quad E_3 = E_2 \Big| \begin{pmatrix} 0 & -1 \\ n & 0 \end{pmatrix},$$

where n is the conductor of ξ . Then by (5.6)

$$(5.8) \quad E' = \epsilon(-1) \cdot E_3 \Big| \begin{pmatrix} N_1 N_2 / n & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} N_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N_1 N_2 & 0 \end{pmatrix} = N_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N_1 & 0 \\ 0 & 1 \end{pmatrix},$$

Lemma 5.1(c) shows

$$E_1 = E_{(\bar{\epsilon}_2, \bar{\epsilon}_2)} \Big| \begin{pmatrix} N_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$L(E_1, s) = -\tau(\epsilon_2) L(\bar{\epsilon}_2, s) L(\bar{\epsilon}_1, s-1).$$

Then (b) of the lemma shows

$$\begin{aligned} L(E_2, s) &= -\tau(\bar{\epsilon}_1) \tau(\epsilon_2) L(\epsilon_1 \bar{\epsilon}_2, s) L(\epsilon_1 \bar{\epsilon}_1, s-1) \\ &= \left[\frac{\tau(\bar{\epsilon}_1) \tau(\epsilon_2)}{\tau(\bar{\xi})} \prod_{p|N} (1 - \xi(p) p^{-s}) \prod_{p|N_1} (1 - p^{1-s}) \right] \\ &\quad \cdot (-\tau(\bar{\xi}) L(\xi, s) \zeta(s-1)). \end{aligned}$$

Thus $D(E_2, s) = [\] \cdot D(E_{(\xi,1)}, s)$. Since $\begin{pmatrix} 0 & -1 \\ n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$, (a) of the lemma shows

$$\begin{aligned} D(E_3, s) &= n^{1-s} D\left(E_2 \left| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s\right.\right) = -n^{1-s} D(E_2, 2-s) \\ &= -n^{1-s} \left\{ \frac{\tau(\bar{\varepsilon}_1)\tau(\varepsilon_2)}{\tau(\bar{\xi})} \prod_{p|N} (1 - \xi(p)p^{s-2}) \prod_{p|N_1} (1 - p^{s-1}) \right\} \cdot D(E_{(\xi,1)}, 2-s) \\ &= n^{1-s} \{ \} \cdot D\left(E_{(\xi,1)} \left| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, s\right.\right) \quad (\text{a}) \\ &= n^{1-s} \{ \} \cdot D(E_{(1,\bar{\xi})}, s) \quad (\text{c}). \end{aligned}$$

Hence $L(E_3, s) = -\{ \} \zeta(s) \cdot L(\bar{\xi}, s-1)$. If we combine this with (5.8) we find

$$\begin{aligned} L(E', s) &= \varepsilon_1(-1) \left(\frac{N_1 N_2}{n} \right)^{1-s} L(E_3, s) \\ &= -\varepsilon_1(-1) \left(\frac{N_1 N_2}{n} \right)^{1-s} \\ &\quad \times \left\{ \frac{\tau(\bar{\varepsilon}_1)\tau(\varepsilon_2)}{\tau(\bar{\xi})} \prod_{p|N} (1 - \xi(p)p^{s-2}) \prod_{p|N_1} (1 - p^{s-1}) \right\} \zeta(s) L(\bar{\xi}, s-1). \end{aligned}$$

The formulas (5.4), (5.5) and

$$a_0(E') = -L(E', 0), \quad \nu(N_1) = N_1 \cdot \prod_{p|N_1} (1 - p^{-1}),$$

$$\zeta(0) = -\frac{1}{2}, \quad L(\bar{\xi}, -1) = -\frac{1}{2} \mathbf{B}_2(\bar{\xi})$$

now lead easily to the desired formula. This proves (d) when $N = N_1 N_2$.

To complete the proof of (d) we proceed by induction on N . Let A be a positive integer divisible by $N_1 N_2$ and l a prime. Suppose the result is known for $N = A$. We prove it for $N = Al$. Let $\Gamma = \Gamma_1(A)$, $\Gamma' = \Gamma_1(Al)$.

Consider the two projections

$$\begin{array}{ccc} & X_1(Al) & \\ \pi \swarrow & & \searrow \pi_l \\ X_1(A) & & X_1(A) \end{array}$$

where π is induced by the inclusion $\Gamma' \subseteq \Gamma$ and π_l is induced by the map $\Gamma' \hookrightarrow \Gamma$ given by $\gamma \mapsto \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} l^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. These maps induce maps on cuspidal divisors

$$\begin{array}{ccc} & \mathcal{D}_{\Gamma'} & \\ \pi^* \nearrow & & \nwarrow \pi_l^* \\ \mathcal{D}_{\Gamma} & & \mathcal{D}_{\Gamma} \end{array}$$

The Dirichlet characters $\varepsilon_1, \varepsilon_2$ define a character on $T(A)$ and also one on $T(Al)$ as in (4.13). We denote both of these characters by the same letter, ψ .

LEMMA 5.2. *Let d be a divisor of A/N_1 such that $N_2|d$. Then*

(a)

$$\pi^*(D_{\Gamma,d}(\psi)) = \begin{cases} l \cdot D_{\Gamma',d}(\psi) & \text{if } l \nmid (A/d), \\ l \cdot D_{\Gamma',d}(\psi) + \varepsilon_1(l) D_{\Gamma',ld}(\psi) & \text{if } l \mid (A/d). \end{cases}$$

(b)

$$\pi_l^*(D_{\Gamma,d}(\psi)) = \begin{cases} lD_{\Gamma',ld}(\psi) & \text{if } l|d, \\ lD_{\Gamma',ld}(\psi) + \bar{\epsilon}_2(l)D_{\Gamma',d}(\psi) & \text{if } l \nmid d. \end{cases}$$

PROOF. Clearly each of these divisors is in $\mathcal{D}_{\Gamma'}(\psi)$. We will give the details of (a) only since (b) is similar.

We have

$$\pi^*(D_{\Gamma,d}(\psi)) = \sum c(k) \cdot D_{\Gamma',k}(\psi),$$

where the sum is over all k such that $k|(Al/N_1)$ and $N_2|k$. The coefficients $c(k)$ are given by

$$c(k) = e\left(\pi; \left[\begin{smallmatrix} 1 \\ k \end{smallmatrix}\right]_{\Gamma'}\right) \cdot r\left(\left[\begin{smallmatrix} 1 \\ k \end{smallmatrix}\right]_{\Gamma'}\right),$$

where $e(\pi; [\frac{1}{k}]_{\Gamma'})$ is the ramification index of π at $[\frac{1}{k}]_{\Gamma'}$ and $r([\frac{1}{k}]_{\Gamma'})$ is the coefficient of $[\frac{1}{k}]_{\Gamma'}$ in $D_{\Gamma',d}(\psi)$. Since $r([\frac{1}{k}]_{\Gamma'}) = 0$ unless $(k, A) = d$, we have $c(k) = 0$ if $k \neq d$ and $k \neq dl$. Moreover, $r([\frac{1}{d}]_{\Gamma'}) = 1$ and

$$r\left(\left[\begin{smallmatrix} 1 \\ dl \end{smallmatrix}\right]_{\Gamma'}\right) = \begin{cases} 0 & \text{if } l|(A/d), \\ \epsilon_1(l) & \text{if } l \nmid (A/d). \end{cases}$$

The ramification indices are easily computed using the formula $e(\pi; x) = e_{\Gamma'}(x)/e_{\Gamma}(x)$. We obtain $e(\pi; [\frac{1}{d}]_{\Gamma'}) = l$ and for $l \nmid (A/d)$, $e(\pi; [\frac{1}{dl}]_{\Gamma'}) = 1$. Thus $c(d) = l$ and

$$c(dl) = \begin{cases} 0 & \text{if } l|(A/d), \\ \epsilon_1(l) & \text{if } l \nmid (A/d). \quad \square \end{cases}$$

LEMMA 5.3. *Let d be a divisor of A/N_1 such that $N_2|d$. Let $E \in \mathcal{E}_{\Gamma}^*$ such that $\delta_{\Gamma}(E) = D_{\Gamma,d}(\psi)$. Let*

$$E_1 = \begin{cases} E & \text{if } l|(A/d), \\ E \left| \left(1 - \frac{\epsilon_1(l)}{l} \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \right) \right. & \text{if } l \nmid (A/d), \end{cases}$$

$$E_2 = \begin{cases} E \left| \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \right. & \text{if } l|d, \\ E \left| \left(\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} - \frac{\bar{\epsilon}_2(l)}{l} \right) \right. & \text{if } l \nmid d. \end{cases}$$

Then $E_1, E_2 \in \mathcal{E}_{\Gamma}^*$ and

$$\delta_{\Gamma'}(E_1) = \begin{cases} lD_{\Gamma',d}(\psi) & \text{if } l|A, \\ l \left(1 - \frac{\xi(l)}{l^2} \right) D_{\Gamma',d}(\psi) & \text{if } l \nmid A, \end{cases}$$

and

$$\delta_{\Gamma'}(E_2) = \begin{cases} lD_{\Gamma',ld}(\psi) & \text{if } l|A, \\ l\left(1 - \frac{\xi(l)}{l^2}\right) \cdot D_{\Gamma',ld}(\psi) & \text{if } l \nmid A. \end{cases}$$

PROOF. This is a straightforward calculation using Lemma 5.2. We will give the details only for E_1 .

Suppose $l|A$. If $l|(A/d)$, then $\delta_{\Gamma'}(E_1) = \delta_{\Gamma'}(E) = \pi^*(D_{\Gamma,d}(\psi)) = lD_{\Gamma',d}(\psi)$ by Lemma 5.2(a). On the other hand if $l \nmid (A/d)$, then $l|d$ and

$$\begin{aligned} \delta_{\Gamma'}(E_1) &= \pi^*(D_{\Gamma,d}(\psi)) - \frac{\varepsilon_1(l)}{l} \pi_l^*(D_{\Gamma,d}(\psi)) \\ &= (lD_{\Gamma',d}(\psi) + \varepsilon_1(l)D_{\Gamma',dl}(\psi)) - \frac{\varepsilon_1(l)}{l} (lD_{\Gamma',dl}(\psi)) \\ &= lD_{\Gamma',d}(\psi). \end{aligned}$$

Suppose $l \nmid A$. Then

$$\begin{aligned} \delta_{\Gamma'}(E_1) &= \pi^*(D_{\Gamma,d}(\psi)) - \frac{\varepsilon_1(l)}{l} \pi_l^*(D_{\Gamma,d}(\psi)) \\ &= (lD_{\Gamma',d}(\psi) + \varepsilon_1(l)D_{\Gamma',ld}(\psi)) - \frac{\varepsilon_1(l)}{l} (lD_{\Gamma',ld}(\psi) + \bar{\varepsilon}_2(l)D_{\Gamma',d}(\psi)) \\ &= l\left(1 - \frac{\xi(l)}{l^2}\right) D_{\Gamma',d}(\psi). \quad \square \end{aligned}$$

The proof of Proposition 4.7(d) follows easily by induction from the case $N = N_1N_2$ and Lemma 5.3. \square

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